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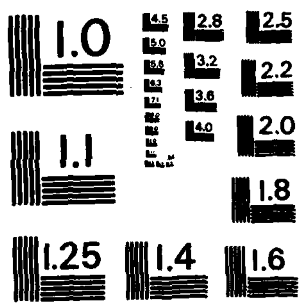

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STRUCTURES REPORT 393

ON THE GENESIS OF RELIABILITY MODELS

by

G. D. MALLINSON

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STRUCTURES REPORT 393

ON THE GENESIS OF RELIABILITY MODELS

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G. D. MALLINSON

SUMMARY

A method for the derivation of reliability models for fatigue is presented and its application demonstrated.

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CONTENTS

1. INTRODUCTION

There are many phenomena (e.g. fatigue, corrosion), which progressively degrade the ability of a structure or component to survive the effects of its service environment. Ultimately this degradation results in the complete failure of the structure at a time which is a function of the environmental and degradation histories. Unfortunately, the uncertainties in both the environmental history and its effect on the degradation process make a prior deterministic calculation of the time of failure impossible.

Certain classes of structures, such as aircraft and automobiles, are manufactured in such a way that for a given environment, the degradation processes are similar between members of each class. It is then possible to apply statistical methods to estimate the time of failure based on a knowledge of the mean behaviour of the class, deduced from experiments or from service failures. Similar statistical methods can be used to account for variations in the environment and/or the effect that the environment has on the degradation process.

A reliability model is one such statistical method. The objective of the model is the computation of reliability functions which predict the behaviour of an ensemble of structures when subjected to sequences of applied loads (the environment) which fall within given statistical descriptions.

There are two approaches which may be used to generate a reliability model for fatigue. The first approach is to assume that strength degradation is related to crack length which is either a known function of time (for a given load application rate)[1–8] or determined by a differential equation.[9–11] Variations in structural behaviour are accommodated by introducing random variables as parameters in the strength–crack length–time relationships.

The second approach is to assume a probabilistic structure from the start and describe the evolution of the probability distributions of critical descriptive random variables using appropriate statistical techniques. An example of this approach is the Markov Chain model described by Bogdanoff and Kozin[12] who also presented a critical appraisal of the two approaches for modelling fatigue crack growth.

The models developed at ARL by Payne *et al.*,[1–4] Hooke[5–8] and Ford[9–11] have used the first approach which has the advantage that, although fatigue is by no means a deterministic process, the ensuing model has the appeal of having a more apparent physical basis.

In parallel with the progressive development of the reliability models at ARL, techniques for their numerical evaluation have been established and refined, culminating in the NERF (*N*umerical *E*valuation of *R*eliability *F*unctions) computer program described by Mallinson and Graham.[13]

The work reported herein originated during the development of the NERF computer program when it became evident that existing reports describing the reliability theory presented neither sufficiently complete analyses nor adequate details of the reliability functions to provide a basis for the specification of the numerical procedures. The computer program was designed for the Payne *et al.* models. It was not clear if the relationship between those models and the analyses presented by Ford[9–11] was sufficiently strong to permit application of the numerical techniques to Ford's models. Moreover, despite similar assumptions, the models of Payne and Hooke were not in complete agreement[8] regarding the expressions for some of the reliability functions.

As this theoretical study progressed, it became evident that a unified method for the derivation of reliability models could successfully yield all the reliability functions developed at ARL and similar functions used elsewhere.[14–15] Differences between the various models could be interpreted and the method fulfilled the original objective of generating a detailed specification for the NERF computer program.[13]

It should be stressed here that the investigation and this report considers only the mathematics behind the derivation of reliability models. It does not consider the problem of obtaining

suitable input data for the model. The various functions and probability density functions used by the reliability analysis are assumed to be known. The estimation of the various parameters required by any model for fatigue is non-trivial and requires very careful analysis of the available data,[16–17] and a complete investigation of the accuracy of such estimations and the effect that any errors may have on the ability of the reliability model to predict a "real life" situation was beyond the scope of this investigation.

1.1 Report Outline

Following the definition of the basic functions evaluated by a reliability model (Chapter 2), the mathematical basis of a unified method for the generation of suitable mathematical expressions for these functions is described (Chapter 3). The analysis indicates how, for any reliability model, the effects of inspections can be included and density functions for any desired structural characteristic obtained. Without making any assumptions regarding the precise form of the model, general results such as the effect of additional but independent hazards to structural survival are deduced. In many cases, it may be advantageous to make a transformation from the original random variables used in the model to new variables to obtain forms of the reliability functions which are easier either to evaluate or to compare with those associated with a different model. The mathematics associated with effecting such transformations are described. Chapter 3 ends with a summary of the steps required to generate a reliability model from the initial assumptions.

The remainder of the report demonstrates the application of the method described in Chapter 3 to derive specific reliability models. Because of the relevance to the documentation of the NERF computer program, the two-parameter model developed by Payne *et al.*[1–4] is considered first (Chapter 4) in some detail. The original random variables used by Payne *et al.* are transformed to new variables representing "age" and "virgin strength". The resulting model which is a little more detailed than the Payne model verifies the functions reported by Payne and Graham.[4] The relevance of approximate expressions derived by Diamond and Payne[3] is considered.

The closely related model developed by Hooke[5–8] is studied in Chapter 5. In terms of the transformed random variables the differences between the models of Payne and Hooke are elucidated. Hooke's proposal to isolate failures by fatigue weakening is examined.

Ford's models[9–11] were developed using methods different from those used by Payne and Hooke. Expressions identical to those presented by Ford[9,11] are generated by the method described here (Chapter 6). Ford's model is extended to include structures that are initially cracked and the similarity between this extended model and the initial crack model proposed by Diamond and Payne[3] is investigated.

A three-parameter model is developed in Chapter 7. This model encompasses all the models described in the previous sections of the report as more simple cases. A tabulation of each model in terms of the three-parameter model is presented.

2. BASIC RELIABILITY FUNCTIONS

The objective of a reliability model is to evaluate "reliability functions" which represent the aggregate behaviour of a population of structures which fall within the scope of a given statistical representation. These basic reliability functions are defined below.

2.1 Risk Rate $r(t)$

The overall effect of the environment and degradation processes is the depletion of the population at a time dependent rate. Defining the removal of a structure from the population as a "failure", this rate can be represented as a risk rate defined in words by,

$$\begin{aligned} r^{*}(t) &= \text{risk rate} \\ &= \frac{\text{No. of failures/(unit time)}}{\text{No. of structures in population at time } t}. \end{aligned} \tag{2.1}$$

The superscript denotes a definition based on a population with a finite number of members. As this number increases, the risk rate approaches the continuous definition,

$$r(t) = \text{risk rate} = \text{Fraction of remaining population failing/(unit time), at time } t. \tag{2.2}$$

2.2 Probability of Failure $P_F(t)$

Let the random variable F denote the time of failure of a structure in the population. At time t, the probability of failure is the probability that F is less than t. This probability can be written as a cumulative distribution $P_F^n(t)$ where,

$$P_F^n(t) = P(F \leqslant t) = \frac{\text{Number of failures up to time } t}{\text{Original number of structures}} \tag{2.3}$$

or for a large population,

$$P_F(t) = \text{Fraction of original population that has failed before time } t. \tag{2.4}$$

$P_F(t)$ ranges from 0 when $t = 0$ to 1 as t becomes infinite.

2.3 Probability of Survival $P_S(t)$

At time t, the probability of survival is the probability that F is greater than t. For a large population,

$$P_S(t) = P(F > t) = \text{Fraction of original population remaining at time } t. \tag{2.5}$$

The total population must, at any time, be composed of two mutually exclusive sets of structures, those that have failed and those that have survived. Obviously,

$$P_F(t)+P_S(t) = 1. \tag{2.6}$$

2.4 Probability Density for the Time to Failure $p_F(t)$

The fraction of the population with times of failure in the interval $(t,t+dt)$ is given by $p_F(t)dt$, where $p_F(t)$ is the probability density function for the time of failure.

2.5 Relationships Between the Basic Reliability Functions

From the definitions given above, relationships between the reliability functions can be derived. The probability distribution of the time to failure is related to $p_F(t)$ by

$$P_F(t) = \int_0^t p_F(t')dt' \tag{2.7}$$

from which it follows that

$$p_F(t) = \frac{dP_F(t)}{dt} = \frac{-dP_S(t)}{dt}. \tag{2.8}$$

For a finite time interval, Δt, the number of failures is

$$p_F(t) \,.\, \Delta t \times (\text{the number of structures in the population}).$$

It follows that,

$$r^n(t) \approx \frac{p_F(t)}{P_S^n(t)}$$

and in the limit as $n \to \infty$,

$$\begin{aligned} r(t) = \frac{p_F(t)}{P_S(t)} &= \frac{dP_F(t)}{dt}\Big/P_S(t) \\ &= \frac{dP_F(t)}{dt}\Big/(1-P_F(t)) \end{aligned} \tag{2.9}$$

or

$$r(t) = \frac{-dP_S(t)}{dt} \Big/ P_S(t). \tag{2.10}$$

Equation (2.10) can be integrated to yield,

$$P_S(t) = \exp\left\{ -\int_0^t r(t)dt \right\}. \tag{2.11}$$

2.6 The Distribution of a Structural Characteristic

Let the random variable $\mathbf{Z}$ denote a particular structural characteristic. The fraction of the population with $z < \mathbf{Z} < z+dz$ failing in the interval $(t,t+dt)$ is given by $p_{\mathbf{F},\mathbf{Z}}(t,z)dt \cdot dz$ where $p_{\mathbf{F},\mathbf{Z}}(t,z)$ is the joint density function for $\mathbf{F}$ and $\mathbf{Z}$. If $p_{\mathbf{F},\mathbf{Z}}(t,z)$ is integrated over all values of z, the total fraction of the population failing in the interval $(t,t+dt)$ is obtained, i.e.,

$$p_{\mathbf{F}}(t) = \int_{-\infty}^{\infty} p_{\mathbf{F},\mathbf{Z}}(t,z)dz. \tag{2.12}$$

The probability density of the time to failure is the marginal density for $\mathbf{F}$.

At a particular value of time, it can be of considerable interest to know the distribution of $\mathbf{Z}$ among the failing structures. This information is obtained by evaluating the conditional density for $\mathbf{Z}$ given failure at time t, $p_{\mathbf{Z}}(z|\mathbf{F} = t)$, written here as $p_{\mathbf{Z}}(z|t)$ where,

$$p_{\mathbf{Z}}(z|t) = \frac{p_{\mathbf{F},\mathbf{Z}}(t,z)}{\int_{-\infty}^{\infty} p_{\mathbf{F},\mathbf{Z}}(t,z)dz}$$

$$= \frac{p_{\mathbf{F},\mathbf{Z}}(t,z)}{p_{\mathbf{F}}(t)} \tag{2.13}$$

Conversely, if $p_{\mathbf{F}}(t)$ can be expressed in the form $\int_{-\infty}^{\infty} f(z)dz$, then,

$$p_{\mathbf{Z}}(z|t) = f(z)/p_{\mathbf{F}}(t). \tag{2.14}$$

The conditional density for $\mathbf{Z}$ given failure at time t will be referred to here as the failure density of the characteristic.

At time t, the fraction of the population with $z < \mathbf{Z} < z+dz$ that has survived is given by $\left[\int_t^{\infty} p_{\mathbf{F},\mathbf{Z}}(t,z)dt\right] dz$. The total fraction surviving to time t is given by the marginal probability of survival,

$$\int_{-\infty}^{\infty} \int_t^{\infty} p_{\mathbf{F},\mathbf{Z}}(t,z)dtdz = P_S(t). \tag{2.15}$$

The conditional density for $\mathbf{Z}$ given survival to time t is $p_{\mathbf{Z}}(z|\mathbf{F} > t)$ where,

$$p_{\mathbf{Z}}(z|\mathbf{F} > t) = \frac{\int_t^{\infty} p_{\mathbf{F},\mathbf{Z}}(t,z)dt}{P_S(t)}. \tag{2.16}$$

Conversely, if $P_s(t)$ can be written in the form $\int_{-\infty}^{\infty} g(z)dz$,

$$p_Z(z|F > t) = g(z)/P_s(t). \tag{2.17}$$

This density function will be referred to here as the density function for the characteristic.

3. MATHEMATICAL BASIS FOR A RELIABILITY MODEL

The derivation of a reliability model for fatigue can be performed by several methods. Payne *et al.*[1-4] derive the aggregate risk rates from first principles. Ford[9,11] derives an equation of continuity for the probability density of crack length and integrates this equation to generate the reliability functions. Hooke[5-8] derives both the probability of survival and the average risk rates from first principles.

The approach used here is to derive the probability of survival directly and then use the relationships established in the previous section to generate the remaining reliability functions. This approach resembles that of Hooke,[5-8] but is more general.

3.1 A General Expression for the Probability of Survival

The derivation of an expression for the probability of survival starts with the following assumptions.

(i) The population of structures can be characterised by a set of M random variables, $X_1, X_2, \ldots, X_M$ with joint probability density function $p_X(\mathbf{x})$ where, to reduce subsequent notation, $\mathbf{X}$ denotes the vector of random variables $X_1, X_2, \ldots, X_M$ and $\mathbf{x}$ denotes the vector of sample values $x_1, x_2, \ldots, x_M$.

(ii) A given structure has a particular combination of values of the random variables. These values remain constant for the life of the structure.

(iii) Each structure passes through a sequence of time zones during each of which a distinct physical process can be represented by an appropriate risk rate that is a function of the random variables and time.

(iv) The number and ordering of the time zones is the same for every member of the population and each structure can fail as a result of only *one* of the physical processes. (The models described herein are strictly one-crack models.)

Expressed formally, for given $\mathbf{x}$ there exists a sequence of times, $t_0(\mathbf{x}), t_1(\mathbf{x}), \ldots, t_K(\mathbf{x})$ such that for $t_{k-1} \leqslant t < t_k$; $(k \leqslant K)$,

$$r(t) = r_k(\mathbf{x},t) \equiv r_k(x_1, x_2, \ldots, x_M, t). \tag{3.1}$$

Conversely, for a given value of time, there exists a set of subspaces $\{D_k\}$ of $\mathbf{x}$ space. In the subspaces D_k

$$r(t) = r_k(\mathbf{x},t).$$

The subspaces are disjoint, i.e. $D_i \bigcap_{i \neq j} D_j = 0$ and $\bigcup_{k=1}^{K} D_k =$ whole of $\mathbf{x}$ space. As time progresses a structure with given $\mathbf{x}$ moves sequentially through the subspaces.

Consider those structures belonging to the M dimensional hypervolume

$$(x_1, x_1+dx_1)\times(x_2, x_2+dx_2)\times\ldots\times(x_M, x_M+dx_M).$$

The fraction of the original population that has these values of the random variables is $p_X(\mathbf{x})d\mathbf{x}$ (where $d\mathbf{x}$ denotes $\prod_{i=1}^{M} dx_i$). As time proceeds, this fraction of the population is depleted by the successive risk rates, $r_k(\mathbf{x},t)$; $1 \leqslant k \leqslant K$. Because, in this volume of $\mathbf{x}$ space the random

variables are essentially constant, equation (2.13) can be used to yield the probability of survival and hence the fraction of this section of the population remaining at time t. For $t_{k-1} < t < t_k$, and denoting the local probability of survival by $P_S^k(\mathbf{x},t)$, the result of this integration is,

$$P_S^k(\mathbf{x},t) = \exp\left\{-\sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} r_j(\mathbf{x},t)dt - \int_{t_{k-1}}^{t} r_k(t)dt\right\}. \quad (3.2)$$

The surviving fraction of this section of the original population is,

$$P_S^k(\mathbf{x},t)p_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \equiv dF_k(\mathbf{x},t), \text{ say.} \quad (3.3)$$

The total fraction of the original population that has survived to time t can be obtained by integrating (3.3) over the whole of $\mathbf{x}$ space, remembering that there are K distinct subspaces to integrate over.

$$\begin{aligned} P_S(t) &= \sum_{k=1}^{K}\left[\int_{D_k} dF_k(\mathbf{x},t)\right] \\ &= \sum_{k=1}^{K}\left[\int_{D_k} \exp\left\{-\sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} r_j(\mathbf{x},t)dt - \int_{t_{k-1}}^{t} r_k(t)dt\right\}p_{\mathbf{X}}(\mathbf{x})d\mathbf{x}\right]. \end{aligned} \quad (3.4)$$

(Note that the integration domain for each D_k is time-dependent.) This expression for the probability of survival is general and can be used to derive a reliability model for any process which satisfies the assumptions made above and for which suitable local (i.e. in terms of the random variables) risk rates can be found.

3.2 A Basis for a Fatigue Model

Hooke[7] and Saunders[17] identify three distinct time zones before failure. In this discussion, two are identified and are considered to be sufficient to provide an adequate model within the scope of available fatigue data. The resulting total of three time zones (including failed structures) is defined below.

(i) D_1: *Uncracked structures*
From initial time (t_0) until the instant of crack initiation, the structure is assumed to be uncracked and its strength the same as when it was new, or in its virgin state. There may be some risk of failure due to service loads being greater than the virgin strength.

(ii) D_2: *Cracked structures*
At time t_1, a fatigue crack is assumed to initiate, following which the structure is weakened and suffers an increased risk that applied loads will exceed its strength. For the purpose of this analysis, crack initiation is assumed to correspond to the time at which strength starts to reduce. The cracked but unweakened phase identified by Hooke is considered to be indistinguishable, from a risk point of view, from the uncracked phase.

During the second time zone, the structure's strength continues to reduce as the fatigue crack grows.

(iii) D_3: *Failed structures*
A frequently adopted assumption of a reliability model is that there is a finite fatigue life for any structure. Thus, in addition to removal from the population as a result of an applied load being greater than the strength of the structure, the population is depleted by structures reaching their fatigue life. This event is often associated with extremely rapid crack growth at $t = t_2$ (Ford) or with a sufficient reduction in the strength of the structure to make it fail under an applied constant load (Payne).

Note that despite the name given to the time zone, D_3 does not contain *all* the structures that have failed; only those that have reached the fatigue time limit.

Clearly, the total fraction of the original population remaining at any time involves integration over D_1 and D_2 only.

To generate a complete model, an expression for the local risk rate in each time zone must be derived. In general, it is in this area that the differences between various fatigue models emerge. There is, however, a certain amount of the analysis that can be made before reaching the model dependent stage.

In the case of fatigue, the risk rate is assumed to be the consequence of an applied fluctuating load sequence which has two major effects.

(i) *Damage assumption*

Each load application produces an increment of damage which, if the structure is uncracked, produces an inevitable approach to crack initiation. Following initiation, the damage becomes an increment in crack length and a subsequent reduction in strength.

(ii) *Contribution to risk*

Each load application has the potential for producing structural failure if the load exceeds the current strength of the structure.

These two effects can be combined to generate a risk function in the following manner. The damage assumption, and the fact that, in terms of the fatigue life of a structure, the applied loads occur with sufficient frequency that variations in their timing may be ignored, permits a direct link between time and load applications to be made via an average load application rate, l_r.

Let R denote the current strength of a structure and let L denote load magnitude. From a given load sequence it is possible to regard L as a random variable and derive a probability density $p_L(L)$ for the applied load. The probability distribution of applied loads is,

$$P_L(L) = \int_0^L p_L(L')dL'. \tag{3.5}$$

Assuming that the applied load sequence is statistically invariant with respect to time, the probability that an applied load exceeds the current strength of a structure is

$$P(L > R) = 1 - P_L(R) = \bar{P}_L(R). \tag{3.6}$$

The instantaneous risk rate is given by

$$r_2(t) = \bar{P}_L(R) \, . \, l_r. \tag{3.7}$$

The damage assumption is included by postulating that the strength of a cracked structure is a function of crack length which is, in turn, a continuous function of the number of load applications. This can be represented formally by,

$$R = R(a(t)) \tag{3.8}$$

where a denotes crack length.

For an uncracked structure the strength is assumed to be constant,* R_0 say, so that,

$$r_1(t) = \bar{P}_L(R_0) \, . \, l_r. \tag{3.9}$$

These expressions for risk pertain, of course, to either a particular structure or to a uniform population of structures. Variations between structures can be accounted for by introducing into equations (3.7)–(3.9) parameters which become the random variables of equation (3.1). It is possible to introduce such parameters in a completely general way thereby accounting for variations in the applied load sequence, crack growth and strength decay. The models that have been used to date, and those described herein, consider statistical variations in the last two mechanisms only. Random variables need only be introduced into the expressions for R, viz.,

$$R_0 = R_0(\mathbf{x}), \tag{3.10}$$

* There is evidence, however, that strength may reduce before crack initiation.[18]

$$R = R(\mathbf{x},t). \tag{3.11}$$

(Note that in general $R_0 \neq R(\mathbf{x},t_0)$.)

To simplify subsequent notation, the following equivalent expressions will be used with the simplest being preferred where possible without ambiguity.

$$r_1 \equiv r_1(\mathbf{x}) \equiv \bar{F}_L(R_0(\mathbf{x})) \,.\, l_r, \tag{3.12}$$

$$r_2 \equiv r_2(\mathbf{x},t) \equiv \bar{F}_L(R(\mathbf{x},t)) \,.\, l_r. \tag{3.13}$$

The boundaries between the three subspaces are defined by the equations,

$$t = t_1(\mathbf{x}) \qquad \text{for } D_1/D_2, \tag{3.14}$$

$$t = t_2(\mathbf{x}) \qquad \text{for } D_2/D_3, \tag{3.15}$$

where D_1/D_2 denotes the boundary between D_1 and D_2, etc.

The analysis leading to equation (3.5) may now be followed through. Equation (3.2) leads to,

$$P_S^1(\mathbf{x},t) = \exp\{-r_1(\mathbf{x}) \,.\, t\}, \tag{3.16}$$

$$P_S^2(\mathbf{x},t) = \exp\left\{-r_1(\mathbf{x}) \,.\, t_1 - \int_{t_1}^{t} r_2(\mathbf{x},t)dt\right\}, \tag{3.17}$$

so that the expression for the probability of survival becomes

$$P_S(t) = \int_{D_1} \exp\{-r_1 t\} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} + \int_{D_2} \exp\left\{-r_1 t_1 - \int_{t_1}^{t} r_2(\mathbf{x},t)dt\right\} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \tag{3.18}$$

A final simplification to the notation can be made by defining a loss factor

$$H(\mathbf{x},t) \equiv H(x_1, x_2, \ldots, x_M, t);\dagger$$

$$H(\mathbf{x},t) = \begin{cases} P_S^1(\mathbf{x},t) & \text{for } t < t_1, \\ P_S^2(\mathbf{x},t) & \text{for } t_1 \leqslant t < t_2, \end{cases} \tag{3.19}$$

which upon replacement in (3.18) yields,

$$P_S(t) = \int_{D_1} H(\mathbf{x},t) p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} + \int_{D_2} H(\mathbf{x},t) p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \tag{3.20}$$

Equations (3.18) or (3.19) and (3.20) define a probability of survival from which all other reliability functions can be generated for a fatigue model.

3.3 The Effects of Inspections

An important requirement of a reliability model is the ability to account for inspections and subsequently provide a basis for the selection of inspection strategies.

An inspection has two effects on the population of structures:

(i) structures that are detected to be deficient according to appropriate criteria are removed;

(ii) the structures removed during an inspection may be replaced.

The first effect is relatively easy to incorporate into the reliability model. Following an inspection at ti_1 say, $p_{\mathbf{X}}(\mathbf{x})$ can be replaced by $p_{\mathbf{X}}^*(\mathbf{x})$ where,

† Note the notational difference between the loss factor defined here, and H, used elsewhere to denote the integrated Hazard $\left(\text{i.e. } \int_0^t r(t)dt\right)$.

$$p_{\mathbf{X}}^{*}(\mathbf{x}) = p_{\mathbf{X}}(\mathbf{x}) \, . \, S(\mathbf{x},ti_1) \tag{3.21}$$

and $S(\mathbf{x},ti_1)$ is a function representing the removal of structures during an inspection. In general, this function can represent both the application of the inspection testing procedures and the probability that those procedures may fail to detect a deficient structure.

The probability of survival following the inspection is,

$$P_S(t) = \int_{D_1} H(\mathbf{x},t)p_{\mathbf{X}}^{*}(\mathbf{x})d\mathbf{x} + \int_{D_2} H(\mathbf{x},t)p_{\mathbf{X}}^{*}(\mathbf{x})d\mathbf{x} \tag{3.22}$$

and the total fraction of the population removed during the inspection, called here the probability of detection at ti_1, $P_{det}(ti_1)$, is

$$P_{det}(ti_1) = P_S^{-}(ti_1) - P_S^{+}(ti_1) \tag{3.23}$$

where $P_S^{-}(ti_1)$ and $P_S^{+}(ti_1)$ denote the probability of survival evaluated by equations (3.20) and (3.22) respectively.

The second effect is more difficult to incorporate. In general, the replacement structures come from a population which has different statistical properties from the original population. At best the replacements may be selected at random from a population similar to the original one. However the replacements may be subjected to repair or inspection procedures which would alter the distributions of the random variables. Accordingly, the probability of survival following the first inspection can be written as,

$$P_S(t) = P^0(t) + P^1(t-ti_1) \, . \, P_{det}(ti_1) \tag{3.24}$$

where $P^0(t)$ represents the probability of survival for the original population (equation (3.22)) and $P^1(t-ti_1)$ represents the probability of survival for the population of replacement structures introduced at time ti_1.

Following a second inspection at ti_2 say, the reliability model must account for three populations; the original population where $p_{\mathbf{X}}^{*}(\mathbf{x})$ is now,

$$p_{\mathbf{X}}^{*}(\mathbf{x}) = p_{\mathbf{X}}(\mathbf{x})S(\mathbf{x},ti_1)S(\mathbf{x},ti_2), \tag{3.25}$$

the ti_1 replacement population, similarly reduced by $S(\mathbf{x},ti_2)$, and a new replacement population of size $P_{det}(ti_2)$. As further inspections occur, the number of populations included in the model continues to increase.

Fortunately, useful calculations can be made without involving the full complexity described above. For example, in the models described by Payne *et al.*, the inspections are assumed to be perfect so that the inspection operator $S(\mathbf{x},t)$ can be replaced by modified integration limits (or subspace boundaries). Replacement structures are assumed to be repaired in such a way that they are no longer susceptible to a risk of failure (Payne[2]) or the calculations may proceed without replacements (Hooke[8]). In either case only one population need be considered in the analysis.

In this report, the inspection analysis developed by Payne *et al.* only will be discussed. A more detailed treatment of the general inspection analysis has been presented by Ford.[10]

3.4 Generation of Derived Functions

Given the expression for the probability of survival, those for the remaining basic reliability functions can be generated by straightforward applications of the relationships (2.6) to (2.11). The general steps involved are outlined in this section for each of the basic functions together with the probability densities for strength and failing load which are defined below and have particular relevance to a fatigue model.

3.4.1 Probability of Failure

Using (2.6),

$$P_F(t) = 1 - P_S(t), \tag{3.26}$$

and the probability of failure can, in principle be obtained in this way. In practice, however, numerical problems arise since a high degree of accuracy in the determination of $P_S(t)$ is required (particularly for low values of t when $P_S(t) \approx 1$), to ensure that a realistic estimate of $P_F(t)$ is made. It is preferable to calculate $P_F(t)$ directly using an expression analogous to (3.20).

Remembering that $P_F^k(\mathbf{x},t) = 1 - P_S^k(\mathbf{x},t)$ so that, in particular, $P_F^3 = 1$, the expression for $P_F(t)$ is,

$$P_F(t) = \int_{D_1} (1-H(\mathbf{x},t))p_{\mathbf{X}}(\mathbf{x})d\mathbf{x} + \int_{D_2} (1-H(\mathbf{x},t))p_{\mathbf{X}}(\mathbf{x})d\mathbf{x} + \int_{D_3} p_{\mathbf{X}}(\mathbf{x})d\mathbf{x}. \quad (3.27)$$

Despite its greater complexity, this expression is in practice numerically easier to evaluate than (3.20). The computer program NERF evaluates $P_F(t)$ this way and derives $P_S(t)$ using (3.26).

3.4.2 Probability Density for the Time to Failure and Risk Rate

The expression for the density function for the time to failure arises from the second relationship in equation (2.8), viz.,

$$p_F(t) = \frac{-dP_S(t)}{dt}. \quad (3.28)$$

Before the details of the differentiation are discussed, it is worth considering the physical meaning of the integral terms in equation (3.20). Each term is an expression for the total fraction of the population contained within a particular subspace and the time differential yields the rate of change of that fraction. With the progression of time, the number of structures in a subspace changes for two reasons. First, the risk incorporated in the loss factor implies a global removal over the subspace. Second, the boundaries of the subspaces are time-dependent, meaning that structures leave or enter the subspace as the boundaries sweep the $\mathbf{x}$ space.

For each subspace, equation (3.28) can be expected to yield three terms; the first representing losses resulting from the applied risk rate, the second and third representing the flow of structures through the boundaries of the subspace. Generally, the second and third terms in a subspace will exactly cancel similar terms from other subspaces. For example, structures crossing the initiation boundary leave D_1 and enter D_2 with no nett loss or gain during the transition. The exception is the term in D_2 which represents removal into the subspace D_3 of failed structures. This term constitutes a direct contribution to $p_F(t)$ and subsequently to $r(t)$. It is the source of the "risk of fatigue fracture" introduced by Payne and Grandage[1] and later questioned by Hooke.[7]

To simplify subsequent analysis, but nevertheless demonstrate the essential mathematics, consider a model which has two independent random variables with density functions $P_{X_1}(x_1)$ and $P_{X_2}(x_2)$. The subspace boundaries are assumed to be such that the term for D_2 in (3.20) can be written in the form,

$$T_2(t) = \int_{x_{2,1}}^{x_{2,2}} \int_{x_{1,1}(t)}^{x_{1,2}(t)} H(x_1,x_2,t)p_{X_1}(x_1)p_{X_2}(x_2)dx_1dx_2 \quad (3.29)$$

where $x_{1,1}$ and $x_{2,1}$ are solutions for equation (3.14), $x_{1,2}$ and $x_{2,2}$ are solutions for equation (3.15) and only $x_{1,1}$ and $x_{1,2}$ are functions of time. The assumption that the limits of the innermost integration are the only limits that are time-dependent is certainly valid for the models considered in this report. The time differential of this term can be written as,

$$\frac{dT_2(t)}{dt} = \int_{x_{2,1}}^{x_{2,2}} \frac{d}{dt}(I_1(x_2,t))p_{X_2}(x_2)dx_2 \quad (3.30)$$

where

$$I_1(x_2,t) = \int_{x_{1,1}(t)}^{x_{1,2}(t)} H(x_1,x_2,t)p_{X_1}(x_1)dx_1. \quad (3.31)$$

Now,

$$\frac{dI_1}{dt} = \int_{x_{1,1}(t)}^{x_{1,2}(t)} \frac{\partial}{\partial t} H(x_1,x_2,t) p_{X_1}(x_1) dx_1 + \frac{dx_{1,2}(t)}{dt} H(x_{1,2},x_2,t) p_{X_1}(x_{1,2}) - $$
$$- \frac{dx_{1,1}(t)}{dt} H(x_{1,1},x_2,t) p_{X_1}(x_{1,1}) \tag{3.32}$$

and

$$\frac{\partial}{\partial t} H(x_1,x_2,t) = -r_k(x_1,x_2,t) H(x_1,x_2,t) \qquad (\text{for } t_{k-1} \leqslant t < t_k) \tag{3.33}$$

so that

$$\frac{dT_2}{dt} = -\int_{x_{2,1}}^{x_{2,2}} \int_{x_{1,1}(t)}^{x_{1,2}(t)} r_2(x_1,x_2,t) H(x_1,x_2,t) p_{X_1}(x_1) p_{X_2}(x_2) dx_1 dx_2 +$$
$$+ \int_{x_{2,1}}^{x_{2,2}} \frac{dx_{1,2}(t)}{dt} H(x_{1,2},x_2,t) p_{X_1}(x_{1,2}) p_{X_2}(x_2) dx_2 -$$
$$- \int_{x_{2,1}}^{x_{2,2}} \frac{dx_{1,1}(t)}{dt} H(x_{1,1},x_2,t) p_{X_1}(x_{1,1}) p_{X_2}(x_2) dx_2. \tag{3.34}$$

By generating a similar expression for dT_1/dt, it can be readily verified that the third term in (3.34) cancels with an identical term representing the flux of structures from D_1 across the crack initiation boundary.

The expression for $p_F(t)$ becomes,

$$p_F(t) = \int_{x_{2,0}}^{x_{2,1}} \int_{x_{1,0}}^{x_{1,1}} r_1(x_1,x_2) H(x_1,x_2,t) p_{X_1}(x_1) p_{X_2}(x_2) dx_1 dx_2 +$$
$$+ \int_{x_{2,1}}^{x_{2,2}} \int_{x_{1,1}}^{x_{1,2}} r_2(x_1,x_2,t) H(x_1,x_2,t) p_{X_1}(x_1) p_{X_2}(x_2) dx_1 dx_2 -$$
$$- \int_{x_{1,1}}^{x_{2,2}} \frac{dx_{1,2}}{dt} H(x_{1,2},x_2,t) p_{X_1}(x_{1,2}) p_{X_2}(x_2) dx_2. \tag{3.35}$$

Noting that $r(t) = p_F(t)P_S(t)$, then $r(t)$ can be written as the sum of three risks,

$$r(t) = r_v(t) + r_s(t) + r_f(t), \tag{3.36}$$

where $r_v(t)$, $r_s(t)$ and $r_f(t)$ correspond to the three terms in equation (3.35) and are named here the virgin risk, the risk of static failure by fatigue and the risk of fatigue life exhaustion. (The second name corresponds with that used by Payne *et al.*; the third corresponds with their "risk of fatigue fracture".)

3.4.3 Failure Density of Strength

At any time it can be of considerable interest to know the magnitude of the loads most likely to cause failure. This information can be obtained by treating the strength, R, as a random variable and, following the analysis of Section 2.6, compute the failure density of strength (also called probability density of the failing load by Diamond and Payne[3]). If the integral expression for $p_F(t)$ can be transformed to the form

$$p_F(t) = \int_0^\infty f(R) dR \tag{3.37}$$

using (3.11), then the required density is,

$$p_R(R|t) = \frac{f(R)}{p_F(t)}. \tag{3.38}$$

3.4.4 Probability Density for Strength

Similarly, the conditional density for **R** given survival to time t can be evaluated by transforming the integral expression for $P_S(t)$ to the form,

$$P_S(t) = \int_0^\infty g(R)dR, \tag{3.39}$$

then

$$p_R(R|\mathbf{F} > t) = \frac{g(R)}{P_S(t)}. \tag{3.40}$$

Examples of the transformations required to generate the densities defined by equations (3.38) and (3.40) are given in Section 4 for the Payne model.

3.5 Some General Results

Before summarising the steps required to generate a reliability model for fatigue, it is worthwhile considering some of the general results that can be deduced from the model equations as they stand, and therefore apply to any of the models derived herein.

3.5.1 The Effect of a Constant Additional Risk

Let

$$r_1'(t) = r_1(\mathbf{x})+r_c, \tag{3.41}$$

$$r_2'(t) = r_2(\mathbf{x},t)+r_c, \tag{3.42}$$

where r_c is a constant risk rate probably resulting from a process different from fatigue, e.g. the hijack risk identified by Ford.[9] Equation (3.2) yields,

$$P_S'^1(\mathbf{x},t) = \exp\{-[r_1(\mathbf{x})+r_c]t\} = P_S^1(\mathbf{x},t)\exp\{-r_ct\} \tag{3.43}$$

and

$$\begin{aligned} P_S'^2(\mathbf{x},t) &= \exp\left\{-r_1(\mathbf{x})t_1-\int_{t_1}^{t} r_2(\mathbf{x},t)dt-r_ct_1-r_c(t-t_1)\right\} \\ &= P_S^2(\mathbf{x},t)\exp\{-r_ct\}. \end{aligned} \tag{3.44}$$

From (3.43) and (3.44) it is clear that,

$$P_S'(t) = \exp\{-r_ct\}P_S(t). \tag{3.45}$$

Now,

$$\frac{d}{dt}P_S'(t) = -r_cP_S'(t)+\exp\{-r_ct\}\frac{d}{dt}P_S(t),$$

so that

$$\begin{aligned} p_F'(t) &= r_c\exp\{-r_ct\}P_S(t)+\exp\{-r_ct\}p_F(t) \\ &= \exp\{-r_ct\}[r_cP_S(t)+p_F(t)] \end{aligned} \tag{3.46}$$

and

$$r'(t) = r_c + p_F(t)/P_S(t)$$
$$= r_c + r(t). \tag{3.47}$$

If

$$P_S(t) = \int_0^\infty g(R)dR,$$

then

$$P_S'(t) = \int_0^\infty g'(R)dR$$

where

$$g'(R) = \exp\{-r_c t\}g(R). \tag{3.48}$$

If

$$p_F(t) = \int_0^\infty f(R)dR,$$

then

$$p_F'(t) = \int_0^\infty f'(R)dR$$

where

$$f'(R) = \exp\{-r_c t\}[g(R)r_c + f(R)]. \tag{3.49}$$

It follows that,

$$p_R'(R|F > t) = \frac{g(R)}{P_S(t)} = p_R(R|F > t) \tag{3.50}$$

and

$$p_R'(R|t) = \frac{r_c g(R) + f(R)}{r_c P_S(t) + p_F(t)}. \tag{3.51}$$

3.5.2 The Effect of a Time Dependent Additional Risk

Let

$$r_1'(t) = r_1(\mathbf{x}) + r_a(t) \tag{3.52}$$

and

$$r_2'(t) = r_2(\mathbf{x},t) + r_a(t) \tag{3.53}$$

where $r_a(t)$ is an additional risk which although time-dependent, is not a function of the random variables. Equation (3.2) yields,

$$P_S'^1(\mathbf{x},t) = \exp\left\{-r_1(\mathbf{x})t - \int_0^t r_a(t)dt\right\}$$
$$= P_S^1(\mathbf{x},t)\exp\left\{-\int_0^t r_a(t)dt\right\} \tag{3.54}$$

and

$$P_S'^2(\mathbf{x},t) = \exp\left\{-r_1(\mathbf{x})t_1 - \int_{t_1}^{t} r_2(\mathbf{x},t)dt - \int_0^{t_1} r_a(t)dt - \int_{t_1}^{t} r_a(t)dt\right\}$$

$$= P_S^2(\mathbf{x},t)\exp\left\{-\int_0^t r_a(t)dt\right\} \tag{3.55}$$

from which,

$$P_S'(t) = \exp\left\{-\int_0^t r_a(t)dt\right\}P_S(t). \tag{3.56}$$

Now,

$$\frac{d}{dt}P_S'(t) = -r_a(t)P'(t)+\exp\left\{-\int_0^t r_a(t)dt\right\}\frac{d}{dt}P_S(t)$$

so that

$$p_F'(t) = \exp\left\{-\int_0^t r_a(t)dt\right\}\left[r_a(t)P_S(t)+p_F(t)\right] \tag{3.57}$$

and

$$r'(t) = r_a(t)+r(t). \tag{3.58}$$

The results for $p_R'(R|\mathbf{F} > t)$ and $p_R'(R|t)$ are identical with equations (3.50) and (3.51) (with r_c replaced by r_a) for the case of constant additional risk.

3.5.3 Transformation of the Random Variables

In some models, it may be convenient, particularly from the viewpoint of their numerical evaluation, to recast the reliability functions in terms of a new set of random variables $\{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_M\}$ which are related to the original set $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M\}$ by an appropriate set of transformation equations.

The easiest transformation to implement is one where each $\mathbf{Y}_i$ is a monotonic function of only one $\mathbf{X}_i$, i.e.,

$$\mathbf{Y}_i = \mathbf{Y}_i(\mathbf{X}_i) \quad \text{and} \quad \mathbf{X}_i = \mathbf{X}_i(\mathbf{Y}_i). \tag{3.59}$$

Consider the integral expression (3.29) for the two-parameter model,

$$T_2(t) = \int_{x_{2,1}}^{x_{2,2}}\int_{x_{1,1}}^{x_{1,2}} H(x_1,x_2,t)p_{\mathbf{X}_1}(x_1)p_{\mathbf{X}_2}(x_2)dx_1dx_2 \tag{3.60}$$

which represents the probability of survival term for the subspace D_2. In terms of the new variables,

$$T_2(t) = \int_{y_{2,1}}^{y_{2,2}}\int_{y_{1,1}}^{y_{1,2}} H_y(y_1,y_2,t)p_{\mathbf{Y}_1}(y_1)p_{\mathbf{Y}_2}(y_2)dy_1dy_2 \tag{3.61}$$

where

$$p_{\mathbf{Y}_i}(y_i) = p_{\mathbf{X}_i}(x_i(y_i))|dx_i/dy_i|, \tag{3.62}$$

$$H_y(y_1,y_2,t) = H(x_1(y_1),x_2(y_2),t) \tag{3.63}$$

and

$$y_{i,j} = y(x_{i,j}). \tag{3.64}$$

For example, if $y_i = c_i x_i$, where c_i is a scaling constant,

$$H_y(y_1,y_2,t) = H(x_1/c_1,x_2/c_2,t), \tag{3.65}$$

$$p_{Y_i}(y_i) = p_{X_i}(y_i/c_i)/c_i \tag{3.66}$$

and

$$y_{i,j} = c_i x_{i,j}. \tag{3.67}$$

A particularly useful transformation can be made by allowing one of the new variables to be time dependent. For example, consider the transformation from $\{X_1, X_2, \ldots, X_M\}$ to $\{Y_1, X_2, \ldots, X_M\}$ such that,

$$Y_1 = Y_1(X_1,t) \qquad X_1 = X_1(Y_1,t) \tag{3.68}$$

and the transformation functions are monotonic. Again, the essential mathematics can be illustrated by considering the D_2 probability of survival term for a two-parameter model satisfying the assumptions for equation (3.30). The transformed expression for $I_1(x_2,t)$, (equation (3.31)), is

$$I_1(x_2,t) = \int_{y_{1,1}(t)}^{y_{1,2}(t)} H_y(y_1,y_2,t) p_{Y_1}(y_1) dy_1 \tag{3.69}$$

where $p_{Y_1}(y_1)$, $H_y(y_1,x_2,t)$ and the integration limits are defined by equations (3.62) to (3.64).

The expression for the D_2 contribution to $p_f(t)$ is generated by taking the time derivative of equation (3.30) which process reduces essentially to establishing dI_1/dt, viz.,

$$\begin{aligned}\frac{dI_1}{dt} = &\int_{y_{1,1}}^{y_{1,2}} \frac{\partial}{\partial t}[H_y(y_1,x_2,t)] p_{Y_1}(y_1) dy_1 + \int_{y_{1,1}}^{y_{1,2}} H_y(y_1,x_2,t) \frac{\partial}{\partial t} p_{Y_1}(y_1) dy_1 + \\ &+ \frac{dy_{1,2}}{dt} p_{Y_1}(y_{1,2}) H_y(y_{1,2},x_2,t) - \\ &- \frac{dy_{1,1}}{dt} p_{Y_1}(y_{1,1}) H_y(y_{1,1},x_2,t)\end{aligned} \tag{3.70}$$

noting that the second term in (3.70) results from the fact that $p_{Y_1}(y_1)$ is time-dependent.

Before proceeding further, it is necessary to examine recasting some of the relationships implied by the transformation equations such as,

$$F(x_1,y_1,t) \equiv y_1 - y_1(x_1,t) \equiv 0. \tag{3.71}$$

Equation (3.71) can be interpreted as an implicit function for x_1 in terms of the independent variables y_1 and t. Now,

$$dF = \frac{\partial F}{dy_1} dy_1 + \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x_1} dx_1$$

and

$$dx_1 = \frac{\partial x_1}{\partial y_1} dy_1 + \frac{\partial x_1}{\partial t} dt$$

which lead to,

$$\left\{\frac{\partial F}{\partial y_1} + \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial y_1}\right\} dy_1 + \left\{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial t}\right\} dt = 0$$

or

$$\left\{1 - \frac{\partial y_1}{\partial x_1}\frac{\partial x_1}{\partial y_1}\right\} dy_1 - \left\{\frac{\partial y_1}{\partial t} + \frac{\partial x_1}{\partial t}\frac{\partial y_1}{\partial t}\right\} dt = 0.$$

Because y_1 and t are independent, the relationships,

$$\frac{\partial y_1}{\partial x_1} = 1\bigg/\frac{\partial x_1}{\partial y_1} = -\frac{\partial y_1}{\partial t}\bigg/\frac{\partial x_1}{\partial t} \tag{3.72}$$

follow.

Returning to the evaluation of dI_1/dt,

$$\begin{aligned}
\frac{\partial}{\partial t}p_{Y_1}(y_1) &= \frac{\partial}{\partial t}\left[p_{X_1}(x_1)\left|\frac{\partial x_1}{\partial y_1}\right|\right] \\
&= \frac{\partial}{\partial x_1}\left[p_{X_1}(x_1)\right]\left|\frac{\partial x_1}{\partial y_1}\right|\frac{\partial x_1}{dt} + p_{X_1}(x_1)\frac{\partial}{\partial y_1}\frac{\partial x_1}{\partial t}\,\mathrm{sign}\left(\frac{\partial x_1}{\partial y_1}\right) \\
&= \frac{\partial}{\partial y_1}\left[p_{Y_1}(y_1)\frac{\partial x_1}{\partial t}\bigg/\frac{\partial x_1}{\partial y_1}\,.\,\mathrm{sign}\left(\frac{\partial x_1}{\partial y_1}\right)\right] \\
&= -\frac{\partial}{\partial y_1}\left[p_{Y_1}(y_1)\frac{\partial y_1}{\partial t}\right] \qquad \text{using (3.72).}
\end{aligned}$$

The second term in (3.70) can be integrated by parts

$$\begin{aligned}
&\int_{y_{1,1}}^{y_{1,2}} H_y(y_1,x_2,t)\frac{\partial}{\partial t}p_{Y_1}(y_1)dy_1 \\
&= -\int_{y_{1,1}}^{y_{1,2}} H_y(y_1,x_2,t)\frac{\partial}{\partial y_1}\left[p_{Y_1}(y_1)\frac{\partial y_1}{\partial t}\right]dy_1 \\
&= -\left[H_y(y_1,x_2,t)p_{Y_1}(y_1)\frac{\partial y_1}{\partial t}\right]_{y_{1,1}}^{y_{1,2}} + \int_{y_{1,1}}^{y_{1,2}}\frac{\partial}{\partial y_1}H_y(y_1,x_2,t)\frac{\partial y_1}{\partial t}p_{Y_1}(y_1)dy_1
\end{aligned}$$

so that (3.70) becomes,

$$\begin{aligned}
\frac{dI_1}{dt} = &\int_{y_{1,1}}^{y_{1,2}}\left\{\frac{\partial}{\partial t}[H_y(y_1,x_2,t)] + \frac{\partial}{\partial y_1}[H_y(y_1,x_2,t)]\frac{\partial y_1}{\partial t}\right\}p_{Y_1}(y_1)dy_1 + \\
&+\left[\frac{dy_{1,2}}{dt} - \left.\frac{\partial y_1}{\partial t}\right|_{y_{1,2}}\right]H_y(y_{1,2},x_2,t)p_{Y_1}(y_{1,2}) - \\
&-\left[\frac{dy_{1,1}}{dt} - \left.\frac{\partial y_1}{dt}\right|_{y_{1,1}}\right]H_y(y_{1,1},x_2,t)p_{Y_1}(y_{1,1})
\end{aligned} \tag{3.73}$$

where the first term reduces finally to $\displaystyle\int_{y_{1,1}}^{y_{1,2}}\frac{d}{dt}[H(y_1,x_2,t)]p_{Y_1}(y_1)dy_1$.

Equation (3.73) can be used to generate the complete expression for $p_Y(t)$, following the analysis leading to (3.35). Examples of the use of transformations to time-dependent random variables appear in the following sections of this report.

3.6 Summary of Steps Required to Generate a Reliability Model for Fatigue

Having detailed the derivation of the reliability functions for a fatigue model, it is worthwhile summarising the essential steps, together with pointers to the relevant equations, required

to generate a reliability model. The model derivations described in the following sections of the report follow the steps described below.

(i) Random variables representing parameters in the fatigue process are defined and corresponding density functions established.

(ii) The various time zones in the fatigue process are identified.

(iii) For each time zone a relationship between risk rate and the random variables is established, equations (3.5)–(3.13).

(iv) Boundaries between the time zones are delineated in terms of the random variables, equations (3.14), (3.15).

(v) If desired, transformed random variables are defined and the equations established to this stage recast in terms of those variables.

(vi) An integral expression for $P_s(t)$ is obtained following the analysis defined by equations (3.16)–(3.22).

(vii) The integral expression for $P_s(t)$ is modified to include the effects of inspections. For the models considered here, the modifications amount to changes in the positions of some of the time zone boundaries.

(viii) $P_F(t)$ is defined using equation (3.27).

(ix) $p_f(t)$ and $r(t)$ are generated by taking the time differential of $P_s(t)$. This follows the example analysis of equations (3.29)–(3.35), using (3.73) if transformed variables are used.

(x) The integral expressions for $P_s(t)$ and $p_F(t)$ are transformed so that (3.38) and (3.40) can be used to evaluate $p_R(R|t)$ and $p_R(R|F \geqslant t)$ (or corresponding densities for some other structural characteristic).

(xi) The effections of additional risks are included as detailed in Sections 3.5.1 and 3.5.2.

4. THE TWO-PARAMETER MODEL OF PAYNE *ET AL.*

One of the functions of this report is to provide a theoretical basis for the documentation of the computer program called NERF which evaluates the reliability models developed by Payne and various co-authors. It is appropriate that the analysis of the previous sections be applied first to the derivation of one of these models.

The reliability model derived in this section is based on the earliest of the Payne models and relies on similar assumptions. The only significant difference is the treatment of the virgin risk for uncracked structures. The Payne[1-4] analysis derived risk rates for cracked structures only; the risk term for the uncracked time zone was ignored. This term is included in the present analysis for completeness and the Payne *et al.* expressions can be obtained by setting $r_1 = 0$.

The expressions developed here correlate most closely with those presented by Payne and Graham,[4] although reference to the most detailed of the earlier papers (Diamond and Payne[3]) will also be made.

4.1 Assumptions and Model Equations

4.1.1 Random Variables

The model assumes that the variation in the fatigue process can be represented by two random variables.

(i) $\mathbf{X}_1$: *Comparative fatigue life*

Each structure in the population is assumed to have a fatigue life t_f given by

$$t_f = \bar{t}_f x_1 \tag{4.1}$$

where $\bar{t}_f$ is a known median fatigue life. A suitable density function $p_{X_1}(x_1)$ is also known.

The parameter x_1 is, in effect, a time scaling parameter which relates the time history of the structure to a median time scale. For example, if $\bar{t}_i$ is the median time to crack initiation, then the initiation time for a structure with fatigue life t_f is $x_1\bar{t}_i$.

(ii) X_2: *Relative residual strength*

For any structure in the population, the ratio of the strength R at a given crack length to the median strength $\bar{R}$ at the same crack length is assumed to be constant for that structure. This constant is the second random variable in the model

$$x_2 = \frac{R(a)}{\bar{R}(a)}. \tag{4.2}$$

4.1.2 Time Zones and Physical Processes

The three time zones indentified in Section 3.2 apply. The boundaries between the time zones are given by

$$t_1 = x_1\bar{t}_i \qquad \text{for } D_1/D_2 \tag{4.3}$$

and

$$t_2 = x_1\bar{t}_f \qquad \text{for } D_2/D_3. \tag{4.4}$$

An additional condition that all structures have a strength greater than a certain specified minimum value, R_{min}, also applies. Thus structures in D_1 or D_2 must satisfy,

$$R > R_{min}. \tag{4.5}$$

4.1.3 Risk Rate Equations

The relationship between strength, crack length and time is assumed to be known in the mean, i.e.,

$$\bar{R} = \bar{R}(a(\bar{t})) \tag{4.6}$$

where the functions have the general form shown in Figure 4.1. For a structure in D_1 the strength is given by

$$R = x_2\bar{R}_0. \tag{4.7}$$

For a structure in D_2 the strength is

$$R = x_2\bar{R}(\bar{a}(t/x_1)). \tag{4.8}$$

Defining

$$\psi(t/x_1) = \bar{R}(a(t/x_1))/\bar{R}_0 \tag{4.9}$$

as the median relative strength decay function,

$$R = x_2\bar{R}_0\psi(t/x_1). \tag{4.10}$$

The risk rates for D_1 and D are,

$$r_1 = r_1(x_2\bar{R}_0) \equiv \bar{P}_L(x_2\bar{R}_0)\,.\,l_r \tag{4.11}$$

$$r_2 = r_2(x_2\bar{R}_0\psi(t/x_1)) \equiv \bar{P}_L(x_2\bar{R}_0\psi(t/x_1))\,.\,l_r. \tag{4.12}$$

4.1.4 Subspace Boundaries

The subspace boundaries can now be defined in terms of the values of the random variables. For structures in D_1,

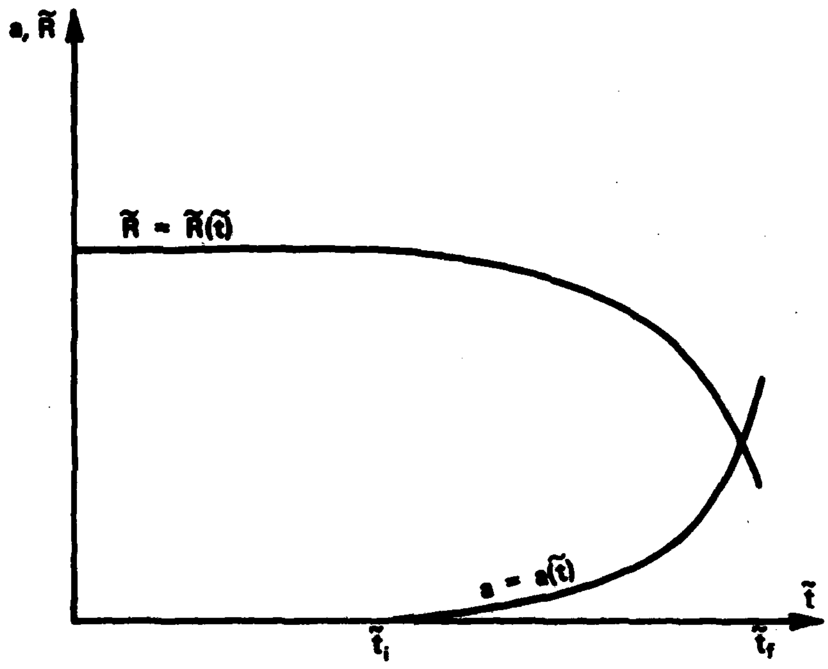

FIG. 4.1. GENERAL FORMS FOR THE STRENGTH AND CRACK LENGTH FUNCTIONS OF TIME.

$$x_2 \geqslant R_{min}/\bar{R}_0 \quad \text{and} \quad t/\bar{t}_i < x_1 < \infty. \tag{4.13}$$

For structures in D_2,

$$x_2 \geqslant R_{min}/\bar{R}_0 \quad \text{and} \quad t/\bar{t}_{t,x_2} < x_1 \leqslant t/\bar{t}_i \tag{4.14}$$

where

$$\bar{t}_{t,x_2} = \min\{\bar{t}_t, \psi^{-1}(R_{min}/\bar{R}_0 x_2)\}, \tag{4.15}$$

or in the reverse order,

$$t/\bar{t}_t < x_1 \leqslant t/\bar{t}_i \quad \text{and} \quad R_{min}/(\bar{R}_0\psi(t/x_1)) \leqslant x_2 < \infty. \tag{4.16}$$

The positions of these boundaries in **x** space are illustrated schematically in Figure 4.2.

4.2 Transformations of the Random Variables

It is convenient to recast the model in terms of two new random variables.

(i) $\mathbf{Y}_1$: *Age*

$$\mathbf{Y}_1 = t/\mathbf{X}_1. \tag{4.17}$$

Whereas all structures with a given value of $\mathbf{X}_1$ have the same fatigue life, at time t structures with the same value of $\mathbf{Y}_1$ have the same value of relative strength decay. In this way $\mathbf{Y}_1$ can be interpreted as a measure of the age of the structure. The probability density for $\mathbf{Y}_1$ is,

$$p_{\mathbf{Y}_1}(y_1) = p_{\mathbf{X}_1}(t/y_1)t/y_1^2. \tag{4.18}$$

(ii) $\mathbf{Y}_2$: *Virgin strength*

$$\mathbf{Y}_2 = \mathbf{X}_2\bar{R}_0. \tag{4.19}$$

All structures with a given value of $\mathbf{Y}_2$ have the same strength before crack *initiation*. This strength is called here the virgin strength. The probability density for $\mathbf{Y}_2$ is,

$$p_{\mathbf{Y}_2}(y_2) = p_{\mathbf{X}_2}(y_2/\bar{R}_0)/\bar{R}_0. \tag{4.20}$$

In terms of the transformed random variables, the risk rate equations are,

$$r_1 = r_1(y_2) \tag{4.21}$$

and

$$r_2 = r_2(y_2\psi(y_1)). \tag{4.22}$$

For structures in D_1,

$$R_{min} \leqslant y_2 \quad \text{and} \quad y_1 < \bar{t}_i \tag{4.23}$$

and for structures in D_2, either

$$R_{min} \leqslant y_2 \quad \text{and} \quad \bar{t}_i \leqslant y_1 \leqslant \bar{t}_{t,y_2} \tag{4.24}$$

where

$$\bar{t}_{t,y_2} = \min\{\bar{t}_t, \psi^{-1}(R_{min}/y_2)\}, \tag{4.25}$$

or

$$\bar{t}_i \leqslant y_1 < \bar{t}_t \quad \text{and} \quad R_{min}/\psi(y_1) \leqslant y_2. \tag{4.26}$$

The positions of the boundaries in **y** space are illustrated schematically in Figure 4.3.

4.3 An Integral Expression for $P_s(t)$

Using (3.2),

$$P_s^1(\mathbf{x},t) = \exp\{-r_1(x_2\bar{R}_0)t\}, \tag{4.27}$$

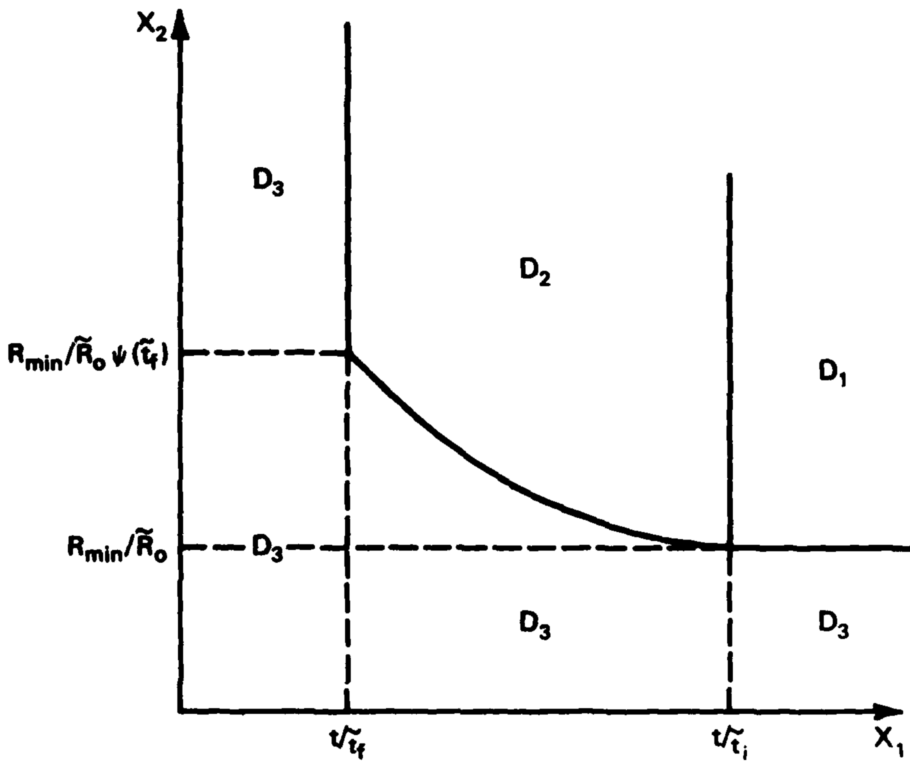

FIG. 4.2. POSITIONS OF SUBSPACE BOUNDARIES IN (X_1, X_2) SPACE.

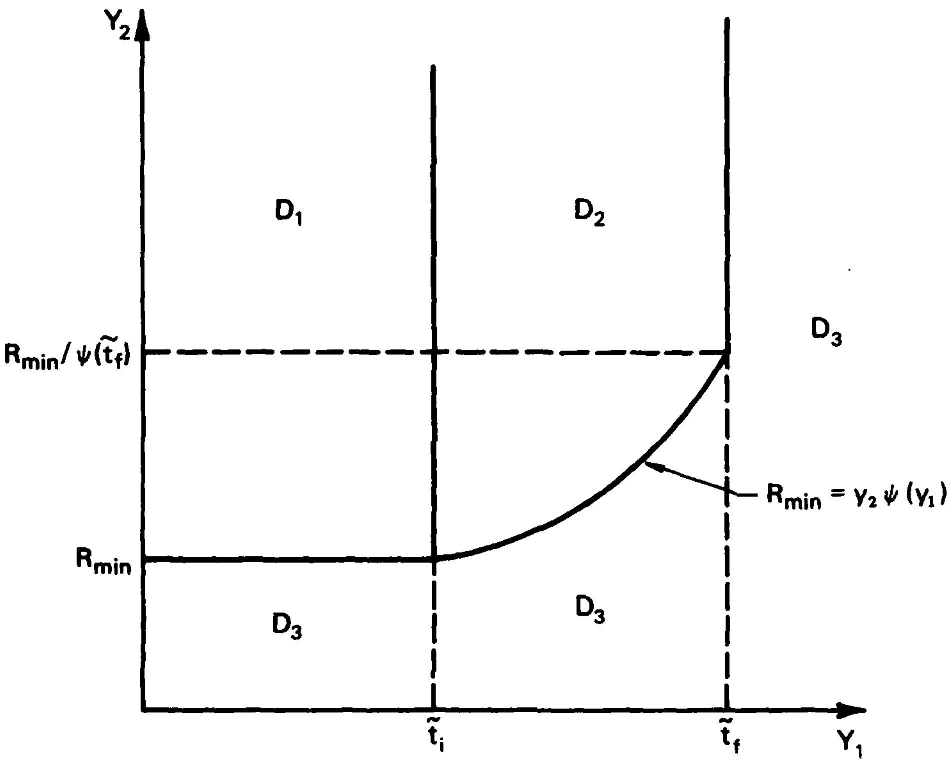

FIG. 4.3. POSITIONS OF SUBSPACE BOUNDARIES IN (Y_1, Y_2) SPACE.

$$P_S^2(\mathbf{x},t) = \exp\left\{-r_1(x_2\bar{R}_0)t_1 - \int_{t_1}^{t} r_2(x_2\bar{R}_0\psi(t'/x_1)dt'\right\} \tag{4.28}$$

or in terms of y,

$$P_S^1(\mathbf{y},t) = \exp\{-r_1(y_2)t\} \tag{4.29}$$

$$P_S^2(\mathbf{y},t) = \exp\left\{-r_1(y_2)t_1 - \int_{t_1}^{t} r_2(y_2\psi(t'y_1/t)dt'\right\}. \tag{4.30}$$

The integration with respect to t' can be transformed by letting

$$y_1' = t'y_1/t$$

so that

$$dt' = (t/y_1)dy_1', \qquad t' = t \Rightarrow y_1' = y_1$$

and

$$t' = t_1 = \bar{t}_1 t/y_1 \Rightarrow y_1' = \bar{t}_1.$$

The expression for $P_S^2(\mathbf{y},t)$ is,

$$\begin{aligned} P_S^2(\mathbf{y},t) &= \exp\left\{-r_1(y_2)\bar{t}_1 t/y_1 - (t/y_1)\int_{\bar{t}_1}^{y_1} r_2(y_2\psi(y_1'))dy_1'\right\} \\ &= \exp\left\{-(t/y_1)\left[r_1(y_2)\bar{t}_1 + \int_{\bar{t}_1}^{y_1} r_2(y_2\psi(y_1'))dy_1'\right]\right\}. \end{aligned} \tag{4.31}$$

Note that the term in square brackets in (4.31) is independent of time. This fact is used to advantage during the numerical evaluation of the reliability functions by the NERF computer program and is a significant motivation for the transformation from X to Y.

The integral expression for $P_S(t)$ can now be constructed using equation (3.20) and the limits defined by (4.23)–(4.26),

$$\begin{aligned} P_S(t) = &\int_{R_{min}}^{\infty} p_{Y_2}(y_2)\exp\{-r_1(y_2)t\}dy_2 P_{Y_1}(\bar{t}_1) + \\ &+ \int_{R_{min}}^{\infty} p_{Y_2}(y_2)\int_{\bar{t}_1}^{\bar{t}_{t,y_2}} \exp\left\{-(t/y_1)\left[r_1(y_2)\bar{t}_1 + \int_{\bar{t}_1}^{y_1} r_2(y_2\psi(y_1'))dy'\right]\right\} p_{Y_1}(y_1)dy_1 dy_2 \end{aligned} \tag{4.32}$$

4.4 Inspections

The early models developed by Payne *et al.* assume that inspection procedures detect deficient structures according to a crack length criterion. The inspections are assumed to be perfect so that all structures with cracks greater than the criterion, denoted here by a_d, are removed from the population. In terms of the reliability model, this means that at the moment of inspection, all structures that have aged to $\bar{t}_d$ where

$$\bar{t}_d = a^{-1}(a_d) \tag{4.33}$$

are removed. For the jth inspection at ti_j say, all structures with $x_1 < ti_j/\bar{t}_d$ are removed so that the effect of the inspection can be represented in equation (3.21) by the function,

$$S(x,ti_j) = H_s(x_1 - ti_j/\bar{t}_d) \tag{4.34}$$

where $H_s(x)$ is the unit step function such that $H_s(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$. The corresponding function for y_1 when $t > ti_j$ is,

$$S(y,ti_j) = H_s(t/y_1 - ti_j/\bar{l}_d). \tag{4.35}$$

All structures with $y_1 > \bar{l}_d t/ti_j$ have been removed from the population.

The effect of an inspection at ti_j can be incorporated in (4.32) by adjusting the upper limit of integration for y_1. This is achieved by replacing $\bar{l}_t$ in equation (4.25) by $\bar{l}_t^*$ where

$$\bar{l}_t^* = \min\{\bar{l}_t, \bar{l}_d t/ti_j\}. \tag{4.36}$$

If the inspections occur very frequently, then the removal of structures may be regarded as being continuous. For this case,

$$\bar{l}_t^* = \min\{\bar{l}_t, \bar{l}_d\}. \tag{4.37}$$

For the remainder of this section the upper limit for the y_1 integration will be assumed to incorporate the effect of inspections prior to the time that the integral expression is being evaluated.

4.5 Generation of $p_F(t)$ and Risk Rates

Equation (4.32) can now be differentiated to obtain the expression for $p_F(t)$. Writing

$$P_s(t) = T_1(t) + T_2(t)$$

and noting that

$$\frac{\partial y_1}{\partial t} = \frac{1}{x_1} = \frac{y_1}{t},$$

$$\frac{dT_1}{dt} = -\int_{R_{min}}^{\infty} p_{Y_2}(y_2) r_1(y_2) \exp\{-r_1(y_2)t\} dy_2 \mathrm{P}_{Y_1}(\bar{l}_i) -$$

$$-\int_{R_{min}}^{\infty} p_{Y_2}(y_2) \exp\{-r_1(y_2)t\} dy_2 \,.\, p_{Y_1}(\bar{l}_i)\bar{l}_i/t, \tag{4.38}$$

where (3.73) has been used to obtain the second term. $T_2(t)$ can be written as,

$$T_2(t) = \int_{R_{min}}^{\infty} p_{Y_2}(y_2) \int_{\bar{l}_i}^{\bar{l}_{t,y_2}} H_y(y_1, y_2, t) p_{Y_1}(y_1) dy_1$$

where

$$H_y(y_1, y_2, t) = \exp\left\{-(t/y_1)\left[r_1(y_2)\bar{l}_i + \int_{\bar{l}_i}^{y_1} r_2(y_2\psi(y_1')) dy_1'\right]\right\}. \tag{4.39}$$

The results,

$$\frac{dH_y}{dt} = -r_2(y_2\psi(y_1)) H_y(y_1, y_2, t),$$

$$H_y(\bar{l}_{t,y_2}, y_2, t) = \exp\left\{-(t/\bar{l}_{t,y_2})\left[r_1(y_2)\bar{l}_i + \int_{\bar{l}_i}^{\bar{l}_{t,y_2}} r_2(y_2\psi(y_1')) dy_1'\right]\right\}$$

and

$$H_y(t_i, y_2, t) = \exp\{-r_2(y_2)t\}$$

may be used to yield,

$$\frac{dT_2}{dt} = -\int_{R_{min}}^{\infty} p_{Y_2}(y_2) \int_{\bar{l}_i}^{\bar{l}_{t,y_2}} r_2(y_2\psi(y_1)) \exp\left\{-(t/y_1)\left[r_1(y_2)\bar{l}_i + \int_{\bar{l}_i}^{y_1} r_2(y_2\psi(y_1')) dy_1'\right]\right\} \ldots \times$$

$$\times p_{Y_1}(y_1) dy_1 dy_2 -$$

$$-\int_{R_{min}}^{\infty} p_{Y_2}(y_2)\frac{\bar{t}_{f,y_2}}{t}\exp\left\{-(t/\bar{t}_{f,y_2})\left[r_1(y_2)\bar{t}_i+\int_{\bar{t}_i}^{\bar{t}_{f,y_2}} r_2(y_2\psi(y_1'))dy_1'\right]\right\}p_{Y_1}(\bar{t}_{f,y_2})dy_2+$$

$$+\int_{R_{min}}^{\infty} p_{Y_2}(y_2)\exp\{-r_1(y_2)t\}(\bar{t}_i/t)p_{Y_1}(\bar{t}_i)dy_2. \qquad (4.40)$$

As expected, the second term for dT_1/dt cancels with a similar flux term for dT_2/dt.

The second term in (4.40) represents the flux of structures from the D_2 subspace and can be decomposed in the following manner. Equation (4.25) implies the following relationships.

$$\bar{t}_{f,y_2} = \psi^{-1}(R_{min}/y_2) = \bar{t}_R \text{ say, for } R_{min} \leqslant y_2 < R_{min}/\psi(\bar{t}_f^*)$$

$$\bar{t}_{f,y_2} = \bar{t}_f^* \text{ for } R_{min}/\psi(\bar{t}_f^*)y_2 < \infty.$$

The integration over y_2 can be divided into two sub-integrations. The first follows the line $R_{min} = y_2\psi(y_1)$ (Fig. 4.3), and yields the total flux of structures that fail because their strength has fallen to R_{min}. The second integration yields the flux of structures that have reached their fatigue life. In the second case, if

$$\bar{t}_f^* = \bar{t}_d t/ti_j < \bar{t}_f,$$

then the factor $\left(\frac{dy_{1,2}}{dt} - \frac{\partial y_i}{\partial t}\Big|_{y_{1,2}}\right)$ in (3.73) is zero so that there is no contribution to $p_F(t)$ from this sub-integration. This corresponds, of course, to the condition that a previous inspection has removed those structures that would have reached their fatigue life limit by the time that $p_F(t)$ is being evaluated.

Bearing the above considerations in mind, the expression for $p_F(t)$ becomes,

$$p_F(t) = \int_{R_{min}}^{\infty} p_{Y_2}(y_2)r_1(y_2)\exp\{-r_1(y_2)t\}dy_2\mathrm{P}_{Y_1}(\bar{t}_i)+$$
$$+\int_{R_{min}}^{\infty} p_{Y_2}(y_2)\int_{\bar{t}_i}^{\bar{t}_{f,y_2}} r_2(y_2\psi(y_1))\exp\left\{-(t/y_1)\left[r_1(y_2)\bar{t}_i+\right.\right.$$
$$\left.\left.+\int_{\bar{t}_i}^{y_1} r_2(y_2\psi(y_1'))dy_1'\right]\right\}p_{Y_1}(y_1)dy_1dy_2+\int_{R_{min}}^{R_{min}/\psi(\bar{t}_f^*)} p_{Y_2}(y_2)\frac{\bar{t}_R}{t}\exp\left\{-(t/\bar{t}_R)\left[r_1(y_2)\bar{t}_i+\right.\right.$$
$$\left.\left.+\int_{\bar{t}_i}^{\bar{t}_R} r_2(y_2\psi(y_1'))dy_1'\right]\right\}p_{Y_1}(\bar{t}_R)dy_2+$$
$$+\delta(\bar{t}_f,\bar{t}_f^*)\int_{R_{min}/\psi(\bar{t}_f)}^{\infty} p_{Y_2}(y_2)\frac{\bar{t}_f}{t}\exp\left\{-(t/\bar{t}_f)\left[r_1(y_2)\bar{t}_i+\right.\right.$$
$$\left.\left.+\int_{\bar{t}_i}^{\bar{t}_f} r_2(y_2\psi(y_1'))dy_1'\right]\right\}p_Y(\bar{t}_f)dy_2. \qquad (4.41)†$$

The total risk $(r(t)=p_F(t)/P_S(t))$, can be written as the sum of three risk terms as in equation (3.36). These risk terms are:

$$r_v(t) = \int_{R_{min}}^{\infty} p_{Y_2}(y_2)r_1(y_2)\exp\{-r_1(y_2)t\}dy_2\int_0^{\bar{t}_i} p_{Y_1}(y_1)dy_1/P_S(t), \qquad (4.42)$$

† $\delta(x,y) = 0 \quad x \neq y,$
$\quad = 1 \quad x = y.$

$$r_s(t) = \int_{R_{min}}^{\infty} p_{Y_2}(y_2) \int_{\tilde{t}_i}^{\tilde{t}_{f,y_2}} r_2(y_2\psi(y_1))H_y(y_1,y_2,t)p_{Y_1}(y_1)dy_1dy_2/P_S(t) \tag{4.43}$$

and

$$r_f(t) = \int_{R_{min}}^{R_{min}/\psi(\tilde{t}_f^*)} p_{Y_2}(y_2)\frac{\tilde{t}_R}{t}H_y(\tilde{t}_R,y_2,t)p_{Y_1}(\tilde{t}_R)dy_2/P_S(t) +$$

$$+\delta(\tilde{t}_f,\tilde{t}_f^*)\int_{R_{min}/\psi(\tilde{t}_f)}^{\infty} p_{Y_2}(y_2)\frac{\tilde{t}_f}{t}H_y(\tilde{t}_f,y_2,t)p_{Y_1}(\tilde{t}_f)dy_2/P_S(t). \tag{4.44}$$

The computer program NERF evaluates the expressions given above for $p_F(t)$ and the three risk rates. For direct comparison with the model developed by Payne *et al.*,[1-4] the risk of failure of uncracked structures is neglected by setting r_1 to zero. The integral expressions for r_s and r_f, in terms of the original random variables x_1 and x_2, are,

$$r_s(t) = \int_{R_{min}/\bar{R}_0}^{\infty} p_{X_2}(x_2) \int_{t/\tilde{t}_{f,x_2}}^{t/\tilde{t}_i} r_2(\bar{R}_0x_2\psi(t/x_1))$$

$$\exp\left\{-\int_{x_1\tilde{t}_i}^{t} r_2(R_0x_2\psi(t'/x_1))dt'\right\}\frac{p_{X_1}(x_1)dx_1dx_2}{P_S(t)} \tag{4.45}$$

and

$$r_f(t) = \int_{R_{min}/\bar{R}_0}^{\infty} p_{X_2}(x_2)\frac{1}{\tilde{t}_{f,x_2}}$$

$$\exp\left\{-\int_{\tilde{t}_it/\tilde{t}_{f,x_2}}^{t} r_2(R_0x_2\psi(t'\tilde{t}_{f,x_2}/t))dt'\right\}p_{X_1}(\tilde{t}_{f,x_2})\frac{dx_2}{P_S(t)} \tag{4.46}$$

Equations (4.45) and (4.46) compare directly with equations (10) and (11) in Payne and Graham.[4] The only difference between the expressions given above and those developed by Payne and Graham is that the lower limit for the x_2 integration is set at $R_{min}/\bar{R}_0$ here to correspond to the known limit for surviving structures. Payne and Graham do not identify the two sub-integrations for the r_f term and attribute all fatigue fracture failures to failures with $R = R_{min}$. Noting that,

$$\tilde{t}_{f,x_2} = \min\{\tilde{t}_f^*,\psi^{-1}(R_{min}/\bar{R}_0x_2)\}, \tag{4.47}$$

their analysis requires that $\tilde{t}_f > \psi^{-1}(R_{min}/\bar{R}_0x_{2max})$ where x_{2max} is the numerical value of x_2 beyond which the density function $p_{X_2}(x_2)$ is insignificant.

This means that the strength function (3.8) must be known for strength values lower than R_{min}. If, as is often the case, there are insufficient experimental data to define the strength function for very low values of strength, then it may be preferable to truncate the function and impose a limit on fatigue life via $\tilde{t}_f$. The two r_f terms in (4.41) will then exist and may be used to determine the contribution made by the poorly defined section of the input data to the total risk rate.

4.6 Probability Density for Strength

The integral expression for $P_S(t)$ can be transformed from the random variables $\mathbf{Y}_1$, $\mathbf{Y}_2$ to $\mathbf{Y}_1$, $\mathbf{R}$ where

$$\mathbf{Y}_2 = \mathbf{R}/\psi(\mathbf{Y}_1) \tag{4.48}$$

and

$$\frac{\partial(y_1,y_2)}{\partial(y_1,R)} = \frac{1}{\psi(y_1)}$$

so that (4.32) becomes,

$$P_S(t) = \int_{R_{\min}}^{\infty} p_{Y_2}(R)\exp\{-r_1(R)t\}dRP_{Y_1}(\bar{l}_i)+$$
$$+\int_{R_{\min}}^{\infty}\int_{\bar{l}_i}^{\bar{l}_f^*} p_{Y_2}(R/\psi(y_1))H_y(y_1,R/\psi(y_1),t)\frac{p_{Y_1}(y_1)}{\psi(y_1)}dy_1dR. \tag{4.49}$$

Defining the strength, R, as a structural characteristic of interest, equation (2.17) can be used to generate the conditional density of strength given survival to time t,

$$p_R(R|F > t) = \Bigg\{p_{Y_2}(R)\exp\{-r_1(R)t\}P_{Y_1}(\bar{l}_i) +$$
$$+\int_{\bar{l}_i}^{\bar{l}_f^*} p_{Y_2}(R/\psi(y_1))\frac{H_y(y_1,R/\psi(y_1),t)}{\psi(y_1)}p_{Y_1}(y_1)dy_1\Bigg\}\Bigg/P_S(t). \tag{4.50}$$

4.7 Failure Density for Strength

The expression (4.41) for $p_F(t)$ can be transformed, using (4.48), into the form,

$$p_F(t) = \int_{R_{\min}}^{\infty} p_{Y_2}(R)r_1(R)\exp\{-r_1(R)t\}dRP_{Y_1}(\bar{l}_i)+$$
$$+\int_{R_{\min}}^{\infty}\int_{\bar{l}_i}^{\bar{l}_f^*} r_2(R)\frac{H(y_1,R/\psi(y_1),t)}{\psi(y_1)}p_{Y_1}(y_1)p_{Y_2}(R/\psi(y_1))dy_1dy_2+$$
$$+\int_{R_{\min}}^{R_{\min}/\psi(\bar{l}_f^*)} p_{Y_2}(y_2)\frac{\bar{l}_R}{t}H_y(\bar{l}_R,y_2,t)p_{Y_1}(\bar{l}_R)dy_2+$$
$$+\delta(\bar{l}_f,\bar{l}_f^*)\int_{R_{\min}}^{\infty} p_{Y_2}(R/\psi(\bar{l}_f))\frac{H_y(\bar{l}_f,R/\psi(\bar{l}_f),t)}{t\psi(\bar{l}_f)}\bar{l}_fP_{Y_1}(\bar{l}_f)dR \tag{4.51}$$

where the third term, being an integration along the line $y_2\psi(y_1) = R_{\min}$ has not been transformed. The resulting expression for the failure density of strength is,

$$p_F(R|t) = \Bigg\{r_1(R)p_{Y_2}(R)\exp\{-r_1(R)t\}P_{Y_1}(\bar{l}_i)+$$
$$+r_2(R)\int_{\bar{l}_i}^{\bar{l}_f^*}\frac{H_y(y_1,R/\psi(y_1),t)}{\psi(y_1)}p_{Y_1}(y_1)p_{Y_2}(R/\psi(y_1))dy_1+$$
$$+\delta(\bar{l}_f,\bar{l}_f^*)p_{Y_2}(R/\psi(\bar{l}_f))\frac{H_y(\bar{l}_f,R/\psi(\bar{l}_f),t)}{t\psi(\bar{l}_f)}\bar{l}_fp_{Y_1}(\bar{l}_f)+$$
$$+\delta(R,R_{\min})\int_{R_{\min}}^{R_{\min}/\psi(\bar{l}_f^*)} p_{Y_2}(y_2)\frac{\bar{l}_R}{t}H_y(\bar{l}_R,R/\psi(y_1),t)p_{Y_1}(\bar{l}_R)dR\Bigg\}\Bigg/P_F(t) \tag{4.52}$$

The last term (which is obtained from the third term in equation (4.51)), represents the concentrated density due to failures with $R = R_{\min}$.

Diamond and Payne[3] derived an expression for the probability distribution for the load at failure. Because the load at failure is equal to the strength at failure, this distribution is equivalent to the cumulative conditional distribution of strength given failure at t,

$$P_{\bar{r}}(R|t) = \int_0^R p_{\bar{r}}(R|t)dR. \tag{4.53}$$

Their expression accounts only for the losses by static failure (r_2) and makes the approximation (see following section), that the loss factor $H_y(y_1,y_2,t) \approx 1$. From the second term of (4.52),

$$P_{\bar{r}}(R|t) \approx \int_0^R r_2(R) \int_{\bar{t}_i}^{\bar{t}_f^*} p_{Y_1}(y_1) p_{Y_2} \frac{(R/\psi(y_1))}{\psi(y_1)} dy_1 dR / p_{\bar{r}}(t),$$

and transforming back to the original random variables, x_1, x_2, with the additional approximation that $P_S(t) \approx 1$ so that $p_{\bar{r}}(t) \approx r(t)$,

$$p_{\bar{r}}(R|t) \approx \int_{t/\bar{t}_f}^{t/\bar{t}_i} \int_0^{R/(\bar{R}_0\psi(t/x_1))} r_2(\bar{R}_0 x_2 \psi(t/x_1)) p_{X_1}(x_1) p_{X_2}(x_2) dx_1 dx_2 / r(t) \tag{4.54}$$

which is the same as the expression given by Diamond and Payne.

4.8 Approximate Expressions for the Risk Rate

During the initial development of the reliability models numerical difficulties associated with the evaluation of the integral expressions provided a strong motivation for simplifying reliability functions. Accordingly, approximate expressions were derived (e.g. Refs. 1–3), by recognising that, in many practical examples, interest is confined to the early part of the population's time history where $P_S(t) \approx 1$ and only a very small section of the population has aged to its fatigue life limit so that $H \approx 1$. Making these approximations and setting $R_{min} = 0$, the expression for the risk of static fracture by fatigue, $r_s(t)$, becomes

$$r_s(t) \approx \int_0^\infty p_{X_2}(x_2) \int_{t/\bar{t}_f^*}^{t/\bar{t}_i} r_2(\bar{R}_0 x_2 \psi(t/x_1)) p_{X_1}(x_1) dx_1 dx_2 \tag{4.55}$$

which corresponds to equation (3.11) of Diamond and Payne.[3] The risk of fatigue fracture is given by,

$$r_f(t) \approx \int_0^\infty \frac{p_{X_2}(x_2) p_{X_1}(t/\bar{t}_f) dx_2}{\bar{t}_f} \approx p_{X_1}(t/\bar{t}_f)/\bar{t}_f. \tag{4.56}$$

Although this approximation has been used for $r_f(t)$, an alternative was derived by Payne.[2] Here, this approximate expression can be deduced by neglecting the effect of the risk in D_2. If r_2 (and r_1) are set to zero, $H(x_1,x_2,t) = 1$ and the expression for the probability of survival is,

$$P_S(t) = \int_{t/\bar{t}_f}^\infty p_{X_1}(x_1) dx_1$$

from which,

$$\frac{dP_S(t)}{dt} = -p_{X_1}(t/\bar{t}_f)/\bar{t}_f$$

so that,

$$r(t) \, . \, (=r_f(t)) = \frac{p_{X_1}(t/\bar{t}_f)/\bar{t}_f}{\int_{t/\bar{t}_f}^\infty p_{X_1}(x_1) dx_1}. \tag{4.57}$$

This expression is in agreement with that derived by Diamond and Payne,[3] with the exception of the factor $1/\bar{t}_f$ which was omitted in Reference 3.

5. THE TWO-PARAMETER MODEL OF HOOKE

The reliability model developed by Hooke[5-8] uses two random variables and is closely related to the Payne model described in previous sections of this report. In fact, Hooke's model can be considered to be encompassed by equation (4.32) for $P_S(t)$ which is more general than the corresponding expressions derived by Payne or Hooke. However, rather than simply enunciate the Hooke equations directly from equation (4.31), Hooke's model will be derived here directly using the approach summarised in Section 3.6.

5.1 Assumptions and Model Equations

X_1: *Characteristic time.*—Hooke lets an appropriate event define a characteristic time for a structure, and makes the assumption that all events in the history of a structure are related to a median time history by a single definition of characteristic time. This means that x_1 is the same time scaling parameter defined by equation (4.1) for the Payne model.

Y_2: *Virgin strength.*—The second random variable in the Hooke model is the same as the virgin strength parameter defined by equation (4.19), following the definition of relative residual strength, for the Payne model.

(Note that although the virgin strength is a basic, as opposed to a transformed, random variable for the Hook model, it is denoted here by Y_2 to emphasize the equivalence with the transformed variable, Y_2, in the Payne model. As far as is practicable, the notation developed in Section 4 will be used in this section.)

The model developed by Hooke corresponds to one in which the third time zone does not exist. There is no imposed fatigue life limit and no specified value for minimum strength. The boundary between D_1 and D_2 is given by,

$$t_1 = x_1\bar{t}_i. \tag{5.1}$$

The model equations are similar to those described in Section 4.1.3, so that

$$r_1 = r_1(y_2) \equiv \bar{F}_L(y_2) \, . \, l_r \tag{5.2}$$

and

$$r_2 = r_2(y_2\psi(t/x_1)) \equiv \bar{F}_L(y_2\psi(t/x_1)) \, . \, l_r. \tag{5.3}$$

For structures in D_1,

$$0 \leqslant y_2 < \infty \quad \text{and} \quad t/\bar{t}_i < x_1 < \infty \tag{5.4}$$

and for those in D_2

$$0 \leqslant y_2 < \infty \quad \text{and} \quad 0 \leqslant x_1 \leqslant t/\bar{t}_i. \tag{5.5}$$

5.2 An Integral Expression for $P_S(t)$

Using the model equations,

$$P_S^1(x_1,y_2,t) = \exp\{-r_1(y_2)t\} \tag{5.6}$$

and

$$P_S^2(x_1,y_2,t) = \exp\left\{-r_1(y_2)x_1\bar{t}_i - \int_{x_1\bar{t}_i}^{t} r_2(y_2\psi(t'/x_1))dt'\right\}. \tag{5.7}$$

The expression for $P_S(t)$ is,

$$P_S(t) = \int_0^\infty \int_{t/\bar{t}_i}^\infty \exp\{-r_1(y_2)t\} p_{X_1}(x_1) p_{Y_2}(y_2) dx_1 dy_2 +$$

$$+\int_0^\infty \int_0^{t/\bar{t}_i} \exp\left\{-r_1(y_2)x_1\bar{t}_i - \right.$$

$$\left. - \int_{x_1\bar{t}_i}^{t} r_2(y_2\psi(t'/x_1))dt'\right\} p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2. \tag{5.8}$$

Writing,

$$r'(y_2\psi(t/x_1)) = \begin{cases} r_1(y_2), & t < x_1\bar{t}_i \\ r_2(y_2\psi(t/x_1)), & t \geqslant x_1\bar{t}_i, \end{cases} \tag{5.9}$$

the two terms in (5.9) can be combined, to produce

$$P_S(t) = \int_0^\infty \int_0^\infty \exp\left\{-\int_0^t r'(y_2\psi(t'/x_1))dt'\right\} p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2 \tag{5.10}$$

which is identical to the expression obtained by Hooke (equation 17 in Ref. 8). Note that the combined expression does not explicitly identify the contribution made by the failures of uncracked structures but relies on that contribution via the definition of r'.

5.3 Generation of $p_F(t)$ and Risk Rates

Equation (5.10) can be differentiated to obtain the expression for $p_F(t)$,

$$p_F(t) = \int_0^\infty \int_0^\infty r'(y_2\psi(t/x_1))\exp\left\{-\int_0^t r'(y_2\psi(t'/x_1))dt'\right\} p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2 \tag{5.11}$$

The resulting expression for total risk $(p_F(t)/P_S(t))$, is identical to equation (19) obtained by Hooke.[8]

Alternatively, the total risk can be considered to be the sum of two risks representing failures in uncracked and cracked structures. Differentiating (5.8) yields,

$$r_v(t) = \int_0^\infty r_1(y_2)\exp\{-r_1(y_2)t\}p_{Y_2}(y_2)dy_2 \int_{t/\bar{t}_i}^\infty p_{X_1}(x_1)dx_1/P_S(t) \tag{5.12}$$

and

$$r_s(t) = \int_0^\infty \int_0^{t/\bar{t}_i} r_2(y_2\psi(t/x_1))H(x_1,y_2,t)p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2/P_S(t) \tag{5.13}$$

where

$$H(x_1,y_2,t) = \exp\left\{-r_1(y_2)x_1\bar{t}_i - \int_{x_1\bar{t}_i}^{t} r_2(y_2\psi(t'/x_1))dt'\right\}. \tag{5.14}$$

5.4 The Effect of Fatigue Life Limiting

The analysis leading to equation (5.10) and (5.11) assumes that there is no imposed limit on fatigue life. As was done in section 4, the effect of such a limit can be accounted for by defining the boundary between the "cracked" and "failed" time zones by,

$$t_2 = x_1\bar{t}_f. \tag{5.15}$$

For a structure with $X_1 = x_1$, equation (5.15) determines the fatigue life limit, or "point of instantaneous strength decay" as identified by Hooke. The boundaries of the D_2 subspace are now defined by,

$$0 \leqslant y_2 < \infty \qquad \text{and} \qquad t/\bar{t}_t < x_1 \leqslant t/\bar{t}_i \tag{5.16}$$

so that from (5.10)

$$P_s(t) = \int_0^\infty \int_{t/\bar{t}_t}^\infty \exp\left\{-\int_0^t r'(y_2\psi(t'/x_1))dt'\right\} p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2 \tag{5.17}$$

and differentiating (5.17)

$$r(t) = \int_0^\infty \int_{t/\bar{t}_t}^\infty r'(y_2\psi(t/x_1))\exp\left\{-\int_0^t r'(y_2\psi(t'/x_1))dt'\right\} \frac{p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2}{P_s(t)} +$$

$$+\frac{p_{X_1}(t/\bar{t}_t)}{\bar{t}_t}\int_0^\infty p_{Y_2}(y_2)\exp\left\{-\int_0^t r'(y_2\psi(\bar{t}_t t'/t))dt'\right\}\frac{dy_2}{P_s(t)}. \tag{5.18}$$

The second term in (5.18) is the risk of fatigue life exhaustion. The corresponding term obtained by Hooke resembles that in (5.18) but contains errors in the exponential factor. The correct expression in his notation is,*

$$r_f(t) = f_{H_f}(t)\int_0^\infty \exp\left\{-\int_0^t m(U_0\zeta(t\,.\,k_f/H_f)/\bar{U}_0)dt\right\} f(U_0)dU_0/R(t). \tag{5.19}$$

Hooke's consideration of fatigue life limiting concerns only the $r_f(t)$ term and the corresponding expressions for $P_s(t)$ and the first term in (5.18) are not given. If fatigue life limiting is applied and the risk of fatigue life exhaustion is calculated separately, the integration limits in the expressions for $P_s(t)$ and $r(t)$ must explicitly reflect the existence of the boundary between the subspaces of cracked and failed structures to prevent double accumulation of failures.

Returning to the more general model developed in Section 4 as a basis for the discussion of the Payne model, the $r_f(t)$ term obtained here is equivalent to the second term in (4.44) with $R_{min} = 0$. In practice, the motivation for applying a limit on fatigue life is often the result of an inability to adequately define the strength function for low values of R and $\bar{t}_f$ will correspond with a minimum value of strength. It is preferable, therefore, to use the more general model which can be reduced to the Payne or Hooke versions by setting $\bar{t}_f = \infty$ or $R_{min} = 0$ respectively.

5.5 Failures Caused by Fatigue Weakening

Consider a population of structures which do not weaken by fatigue. The only non-empty subspace is D_1 and

$$r' = r_1 = r_1(y_2) = \bar{P}_L(y_2)\,.\,l_r. \tag{5.20}$$

The probability of survival is given by,

$$P_s^u(t) = \int_0^\infty \int_0^\infty \exp\{-r_1(y_2)t\} p_{X_1}(x_1)p_{Y_2}(y_2)dx_1dy_2$$

$$= \int_0^\infty \exp\{-r_1(y_2)t\} p_{Y_2}(y_2)dy_2 \tag{5.21}$$

which is identified here, in line with Hooke, as the survivorship with strength preserved. It follows that the total risk which will be called the risk of ultimate failure, $r_u(t)$, is given by,

$$r_u(t) = \int_0^\infty r_1(y_2)\exp\{-r_1(y_2)t\} p_{Y_2}(y_2)dy_2/P_s^u(t) \tag{5.22}$$

---

* This notation is not defined in the table of notation as it is local to this equation (5.19) only.

Returning to the population in which fatigue weakening is occuring, Hooke identifies failures that have occurred *because* of the fatigue process. Such failures are those of cracked structures by loads that are less than the initial (uncracked) strengths. Hooke then computes the "probability of failure of weakened structures by loads below their virgin strengths", $P_F^w(t)$ say, where

$$P_F^w(t) = P_S^u(t) - P_S(t) \tag{5.23}$$

$P_S^u(t)$ is the probability of survival defined by equation (5.21) and $P_S(t)$ is defined by (5.10) (or (5.17) if fatigue life is limited).

Hooke's approach is, in fact, incorrect. Using (2.6), equation (5.23) becomes,

$$P_F^w(t) = P_F(t) - P_F^u(t) \tag{5.24}$$

which on differentiation with respect to time yields,

$$p_F(t) = p_F^w(t) + p_F^u(t). \tag{5.25}$$

While it is true that $p_F(t)$ can be decomposed into two additive terms, that term representing failures by loads greater than the virgin strengths is not equal to $p_F^u(t)$ (as obtained by multiplying (5.22) by $P_S^u(t)$). Any of the reliability functions derived from equation (5.21) cannot account for the fact that failures contributing to $p_F^w(t)$ reduce the effective population of structures that can fail by loads greater than the virgin strengths. $p_F^u(t)$ will therefore be an over-estimate for the contribution to $p_F(t)$ from failures that have not occurred because of the fatigue process, and (5.23) will over-estimate $P_F^w(t)$.

The correct approach is to introduce the distinction between the two types of failures during the definition of the model equations. This can be done for the Hooke model by replacing equation (5.3) by,

$$r_2 = r_2''(y_2\psi(t/x_1)) + r_1(y_2) \tag{5.26}$$

where

$$r_2''(y_2\psi(t/x_1)) = [\bar{F}_L(y_2\psi(t/x_1)) - \bar{F}_L(y_2)] \cdot \bar{l}_t. \tag{5.27}$$

For structures with $X_1 = x_1$ and $Y_2 = y_2$, r_2'' is the *risk rate for failures* by loads less than the virgin strength, y_2. The expression for the probability of survival (with fatigue life limiting) is,

$$P_S(t) = \int_0^\infty \int_{t/\bar{t}_1}^\infty \exp\{-r_1(y_2)t\} p_{X_1}(x_1) p_{Y_2}(y_2) dx_1 dy_2 + \\ + \int_0^\infty \int_{t/\bar{t}_t}^{t/\bar{t}_1} H(x_1,y_2,t) p_{X_1}(x_1) p_{Y_2}(y_2) dx_1 dy_2 \tag{5.28}$$

where the loss factor in D_2 is now,

$$H(x_1,y_2,t) = \exp\left\{-r_1(y_2)t - \int_{x_1\bar{t}_1}^t r_2''(y_2\psi(t'/x_1))dt'\right\} \\ = \exp\{-r_1(y_2)t\}\exp\left\{-\int_{x_1\bar{t}_1}^t r''(y_2\psi(t'/x_1))dt'\right\}. \tag{5.29}$$

The expression for $p_F(t)$ is,

$$p_F(t) = \int_0^\infty \int_{t/\bar{t}_1}^\infty r_1(y_2)\exp\{-r_1(y_2)t\} p_{X_1}(x_1) p_{Y_2}(y_2) dx_1 dy_2 + \\ + \int_0^\infty p_{Y_2}(y_2) \int_{t/\bar{t}_t}^{t/\bar{t}_1} [r_1(y_2) + r_2''(y_2\psi(t/x_1))] H(x_1,y_2,t) p_{X_1}(x_1) dx_1 dy_2 + \\ + \frac{p_{X_1}(t/\bar{t}_t)}{\bar{t}_t} \int_0^\infty p_{Y_2}(y_2) H(t/\bar{t}_t,y_2,t) dy_2 \tag{5.30}$$

which can be rearranged to yield,

$$
\begin{aligned}
p_{\mathbf{T}}(t) = \int_0^\infty p_{\mathbf{Y}_2}(y_2) r_1(y_2) \exp\{-r_1(y_2)t\} \Bigg[\int_{t/\bar{t}_i}^\infty p_{\mathbf{X}_1}(x_1)dx_1 + \\
+ \int_{t/\bar{t}_f}^{t/\bar{t}_i} \exp\left\{ - \int_{\bar{t}_i x_1}^{t} r_2^w(y_2 \psi(t'/x_1))dt' \right\} p_{\mathbf{X}_1}(x_1)dx_1 \Bigg] dy_2 + \\
+ \int_0^\infty p_{\mathbf{Y}_2}(y_2) \int_{t/\bar{t}_f}^{t/\bar{t}_i} r_2^w(y_2 \psi(t/x_1)) H(x_1, y_2, t) p_{\mathbf{X}_1}(x_1)dx_1 dy_2 + \\
+ \frac{p_{\mathbf{X}_1}(t/\bar{t}_f)}{\bar{t}_f} \int_0^\infty p_{\mathbf{Y}_2}(y_2) H(t/\bar{t}_f, y_2, t)dy_2.
\end{aligned}
\tag{5.31}
$$

The last two terms in (5.31) represent the required density, $p_{\mathbf{T}}^w(t)$, for failures due to fatigue weakening. The first term is the density for failures by loads greater than the virgin strength. The correct version of equation (5.25) for this model is,

$$
\begin{aligned}
p_{\mathbf{T}}^w(t) = p_{\mathbf{T}}(t) - p_{\mathbf{T}}^u(t) + \int_0^\infty p_{\mathbf{Y}_2}(y_2) r_1(y_2) \exp\{-r_1(y_2)t\} \Bigg[1 - \int_0^{t/\bar{t}_f} p_{\mathbf{X}_1}(x_1)dx_1 + \\
+ \int_{t/\bar{t}_f}^{t/\bar{t}_i} \left(1 - \exp\left\{ - \int_{\bar{t}_i x_1}^{t} r_2^w(y_2 \psi(t'/x_1))dt' \right\} \right) p_{\mathbf{X}_1}(x_1)dx_1 \Bigg] dy_2.
\end{aligned}
\tag{5.32}
$$

6. TWO-PARAMETER MODEL OF FORD

Whereas Payne and Hooke derived their reliability models using techniques which bear a close resemblance to those used here, Ford solved a continuity equation for the probability density for crack length to generate reliability functions. The functions for the two-parameter model for "coarsely random cracking" are derived here and provide verification that the two different methods yield the same results.

6.1 Assumptions and Model Equations

The model is based on two random variables.

(i) $\mathbf{X}_1$: *Comparative crack growth rate*
Ford assumes that the equation for crack growth can be written in the form,

$$da/dt = \mathbf{x}_1 f(a) \tag{6.1}$$

with $\mathbf{x}_1 = 1$ being equivalent to the known median equation. Equation (6.1) can be transformed,

$$da/d(t/\mathbf{x}_1) = f(a), \tag{6.2}$$

so that $\mathbf{x}_1$ can be interpreted as a scale factor for time. An event occurring at $\bar{t}-\bar{t}_0$, where $\bar{t}_0$ is some suitable reference time in the median time history, will occur at $(\bar{t}-\bar{t}_0)/x_1$ for a structure with $\mathbf{X}_1 = x_1$. Clearly $\mathbf{X}_1$ is the reciprocal of the fatigue life parameter defined in Section 4 for the Payne model.

(ii) $\mathbf{X}_2$: *Crack initiation time*
An obvious choice for a reference event is crack initiation. If this event is assigned variation and a random variable, $\mathbf{X}_2$, an event occurring at time $\bar{t}$ in the median time history will occur at time t in the history of a structure with $\mathbf{X}_1 = x_1$ and $\mathbf{X}_2 = x_2$ where,

$$t = x_2 + (\bar{t} - \bar{t}_i)/x_1. \tag{6.3}$$

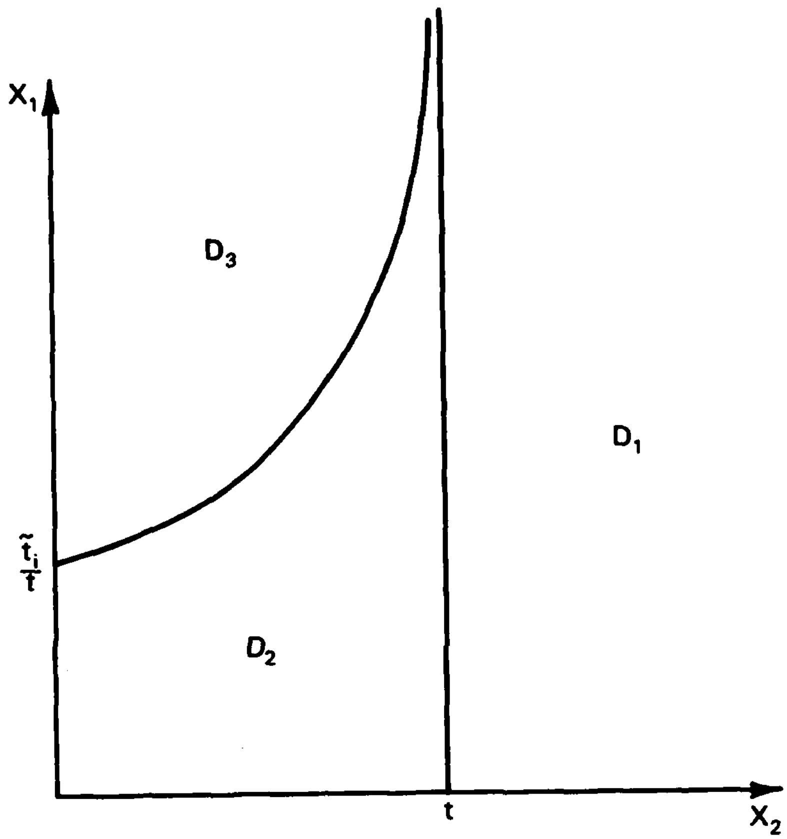

FIG. 6.1. POSITIONS OF SUBSPACE BOUNDARIES IN (X_2,X_1) SPACE FOR THE 2 PARAMETER FORD MODEL.

Ford imposes fatigue life limiting via a runaway crack hypothesis and because there is no variation in strength, a limit on fatigue life defines a corresponding minimum value of strength. The boundaries between the three time zones are,

$$t_1 = x_2 \qquad \text{for } D_1/D_2 \tag{6.4}$$

and

$$\begin{aligned} t_2 &= (\bar{l}_t - \bar{l}_i)/x_1 + x_2 \\ &= \bar{l}_l/x_1 + x_2, \quad \text{say.} \end{aligned} \tag{6.5}$$

Since there is no variation in strength, the risk rate in D_1 is constant and was included by Ford with an independent "hijack" risk in an additional, constant risk term. In this analysis the hijack risk is not included during the development of the model but can be included subsequently using the general results obtained in Section 3.5.1. The model equations are,

$$r_1 = \bar{P}_L(\bar{R}_0) \,.\, l_r \tag{6.6}$$

and

$$r_2 = r_2(x_1(t-x_2)) = \bar{P}_L(\bar{R}_0\psi(\bar{l}_i + x_1(t-x_2))). \tag{6.7}$$

For structures in D_1,

$$t < x_2 < \infty, \qquad 0 < x_1 < \infty \tag{6.8}$$

and for those in D_2

$$0 < x_2 \leqslant t, \qquad 0 < x_1 < \bar{l}_l/(t-x_2). \tag{6.9}$$

The locations of the three subspaces in (x_2, x_1) space are shown in Figure 6.1.

6.2 An Integral Expression for $P_S(t)$

Using the model equations,

$$P_S^1(\mathbf{x},t) = \exp\{-r_1 t\}, \tag{6.10}$$

$$P_S^2(\mathbf{x},t) = \exp\left\{-r_1 x_2 - \int_{x_2}^{t} r_2(x_1(t'-x_2))dt'\right\}, \tag{6.11}$$

and the expression for $P_S(t)$ is,

$$\begin{aligned} P_S(t) &= \int_t^\infty \int_0^\infty \exp\{-r_1 t\} p_{X_1}(x_1) p_{X_2}(x_2) dx_1 dx_2 + \\ &\quad + \int_0^t \int_0^{\bar{l}_l/(t-x_2)} \exp\left\{-r_1 x_2 - \int_{x_2}^{t} r_2(x_1(t-x_2))dt'\right\} p_{X_1}(x_1) p_{X_2}(x_2) dx_1 dx_2 \\ &= \exp\{-r_1 t\}\bar{P}_{X_2}(t) + \\ &\quad + \int_0^t \exp\{-r_1 x_2\} p_{X_2}(x_2) \int_0^{\bar{l}_l/(t-x_2)} \exp\left\{-\int_{x_2}^{t} r_2(x_1(t'-x_2))dt'\right\} p_{X_1}(x_1) dx_1 dx_2. \end{aligned} \tag{6.12}$$

6.3 Calculation of $p_T(t)$ and Risk Rates

Equation (6.12) can be differentiated to generate the expression for the density of the time to failure.

$$
\begin{aligned}
p_{\mathbf{F}}(t) &= r_1 \exp\{-r_1 t\} \bar{P}_{\mathbf{X}_1}(t) + \\
&\quad + \int_0^t \exp\{-r_1 x_2\} p_{\mathbf{X}_2}(x_2) \frac{d}{dt}\left[\int_0^{\bar{l}_1/(t-x_2)} \exp\left\{-\int_{x_2}^{t} r_2(x_1(t'-x_2))dt'\right\} p_{\mathbf{X}_1}(x_1)dx_1\right]dx_2 \\
&= r_1 \exp\{-r_1 t\} \bar{P}_{\mathbf{X}_1}(t) + \\
&\quad + \int_0^t \exp\{-r_1 x_2\} p_{\mathbf{X}_2}(x_2) \phi(t-x_2) dx_2,
\end{aligned}
\tag{6.13}
$$

where

$$
\begin{aligned}
\phi(t-x_2) &= \int_0^{\bar{l}_1/(t-x_2)} r_2(x_1(t-x_2)) \exp\left\{-\int_{x_2}^{t} r_2(x_1(t'-x_2))dt'\right\} p_{\mathbf{X}_1}(x_1)dx_1 + \\
&\quad + \frac{\bar{l}_1}{(t-x_2)^2} \exp\left\{-\int_{x_2}^{t} r_2(\bar{l}_1(t'-x_2)/(t-x_2))dt'\right\} p_{\mathbf{X}_1}(\bar{l}_1/(t-x_2))
\end{aligned}
\tag{6.14}
$$

which by changing the variable of integration in the exponential term becomes,

$$
\begin{aligned}
\phi(t-x_2) &= \int_0^{\bar{l}_1/(t-x_2)} r_2(x_1(t-x_2)) \exp\left\{-\frac{1}{x_1}\int_0^{x_1(t-x_2)} r_2(c)dc\right\} p_{\mathbf{X}_1}(x_1)dx_1 + \\
&\quad + \frac{\bar{l}_1}{(t-x_2)^2} \exp\left\{-\frac{(t-x_2)}{\bar{l}_1}\int_0^{\bar{l}_1} r_2(c)dc\right\} p_{\mathbf{X}_1}(\bar{l}_1/(t-x_2)).
\end{aligned}
\tag{6.15}
$$

Equations (6.13) and (6.15) compare directly with (4.3) and (4.4) presented by Ford.[11] Apart from a small error in Ford's second term of (4.4) arising from an algebraic error in his equation (3.4), the expressions obtained via the two different methods are identical.

The expressions for the three components of the risk rate are,

$$
r_v(t) = r_1 \exp\{-r_1 t\} \bar{P}_{\mathbf{X}_2}(t)/P_{\mathbf{S}}(t), \tag{6.16}
$$

$$
\begin{aligned}
r_s(t) &= \int_0^t \exp\{-r_1 x_2\} p_{\mathbf{X}_2}(x_2) \int_0^{\bar{l}_1/(t-x_2)} r_2(x_1(t-x_2)) \times \\
&\quad \times \exp\left\{-\frac{1}{x_1}\int_0^{x_1(t-x_2)} r_2(c)dc\right\} \frac{p_{\mathbf{X}_1}(x_1)dx_1 dx_2}{P_{\mathbf{S}}(t)}
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
r_t(t) &= \int_0^t \exp\{-r_1 x_2\} p_{\mathbf{X}_2}(x_2) \frac{\bar{l}_1}{(t-x_2)^2} \times \\
&\quad \times \exp\left\{-\frac{(t-x_2)}{\bar{l}_1}\int_0^{\bar{l}_1} r_2(c)dc\right\} \frac{p_{\mathbf{X}_1}(\bar{l}_1/(t-x_2))dx_2}{P_{\mathbf{S}}(t)}.
\end{aligned}
\tag{6.18}
$$

6.4 Transformed Random Variables

Having established the equivalence between the model described by equation (6.12) and that derived by Ford, it is convenient for the purposes of further discussion and development to introduce new random variables which are transformations of the original variables used by Ford. Specifically, the introduction of a random variable representing age will facilitate comparisons with the models of Payne and Hooke.

Accordingly, let

$$
\mathbf{Y}_2 = \mathbf{X}_1 \tag{6.19}
$$

and

$$\mathbf{Y}_1 = \begin{cases} \bar{l}_i t/\mathbf{X}_2, & \mathbf{X}_2 > t \\ \bar{l}_i + \mathbf{Y}_2(t - \mathbf{X}_2), & \mathbf{X}_2 < t \end{cases} \tag{6.20}$$

define a transformation from **X** to **Y** with $\mathbf{Y}_1$ being the required age variable. For $\mathbf{X}_2 \leqslant t$ (or $\mathbf{Y}_1 \geqslant \bar{l}_i$), the random variables $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are not independent; the joint density function $p_{\mathbf{Y}_1,\mathbf{Y}_2}(y_1,y_2)$ is given by,

$$\begin{aligned} p_{\mathbf{Y}_1,\mathbf{Y}_2}(y_1,y_2) &= p_{\mathbf{X}_1}(y_2)p_{\mathbf{X}_2}(t-(y_1-\bar{l}_i)/y_2)/y_2 \\ &\equiv p_{\mathbf{Y}_2}(y_2)p_{\mathbf{Y}_1}(y_1|y_2) \end{aligned} \tag{6.21}$$

where $p_{\mathbf{Y}_1}(y_1|y_2)$ is the conditional density for $\mathbf{Y}_1$ given $\mathbf{Y}_2 = y_2$. For $\mathbf{X}_1 \leqslant t, P_{\mathbf{Y}_1}(y_1)$, the marginal density for $\mathbf{Y}_1$ is given by,

$$p_{\mathbf{Y}_1}(y_1) = \int_0^\infty p_{\mathbf{Y}_2}(y_2)p_{\mathbf{Y}_1}(y_1|y_2)dy_2. \tag{6.22}$$

For $\mathbf{X}_1 > t$,

$$p_{\mathbf{Y}_1}(y_1) = p_{\mathbf{X}_2}(\bar{l}_i t/y_1)\bar{l}_i t/y_1^2. \tag{6.23}$$

In terms of the transformed random variables, the model equations are given by (6.6) for r_1 and

$$r_2 = r_2(y_1) = \bar{P}_L(\bar{K}_0\psi(y_1)). \tag{6.24}$$

(Note the *difference* between equations (6.7) and (6.24) in the definition of the argument for r_2. In (6.24) the argument of r_2 is y_1; in (6.7) the argument is $y_1 - \bar{l}_i$.)

For structures in D_1,

$$0 < y < \bar{l}_i, \tag{6.25}$$

and for those in D_2

$$0 < y_2 < \infty, \qquad \bar{l}_i \leqslant y_1 < \min\{\bar{l}_t, \bar{l}_i + y_2 t\}, \tag{6.26}$$

as illustrated in Figure 6.2. Note that the Ford model does not include structures with $x_1 < 0$ (i.e. structures with pre-existing cracks). This condition restricts the maximum value of y_1 for a given crack growth rate, y_2.

Using equations (6.6) and (6.24),

$$P_S^1(\mathbf{y},t) = \exp\{-r_1 t\}, \tag{6.27}$$

$$\begin{aligned} P_S^2(\mathbf{y},t) &= \exp\left\{-(t-(y_1-\bar{l}_i)/y_2)r_1 - \int_{t-(y_1-\bar{l}_i)/y_2}^{t} r_2(y_1+y_2(t'-t))dt'\right\} \\ &= \exp\left\{-(t-(y_1-\bar{l}_i)/y_2)r_1 - \frac{1}{y_2}\int_{\bar{l}_i}^{y_1} r_2(y_1')dy_1'\right\}. \end{aligned} \tag{6.28}$$

Noting that,

$$\begin{aligned} \int_0^\infty \int_0^{\bar{l}_i} \exp\{-r_1 t\}p_{\mathbf{Y}_1,\mathbf{Y}_2}(y_1,y_2)dy_1dy_2 &= \exp\{-r_1 t\}\int_0^\infty \int_0^{\bar{l}_i} p_{\mathbf{Y}_2}(y_2)p_{\mathbf{Y}_1}(y_1)dy_1dy_2 \\ &= \exp\{-r_1 t\}P_{\mathbf{Y}_1}(\bar{l}_i), \end{aligned}$$

the expression for the probability of survival is,

$$\begin{aligned} P_S(t) = {} & \exp\{-r_1 t\}P_{\mathbf{Y}_1}(\bar{l}_i) + \\ & + \int_0^\infty p_{\mathbf{Y}_2}(y_2)\int_{\bar{l}_i}^{\bar{l}_{t,y_2}} p_{\mathbf{Y}_1}(y_1|y_2)\exp\left\{-(t-(y_1-\bar{l}_i)/y_2)r_1 - \right. \\ & \left. - \frac{1}{y_2}\int_{\bar{l}_i}^{y_1} r_2(y_1')dy_1'\right\}dy_1dy_2 \end{aligned} \tag{6.29}$$

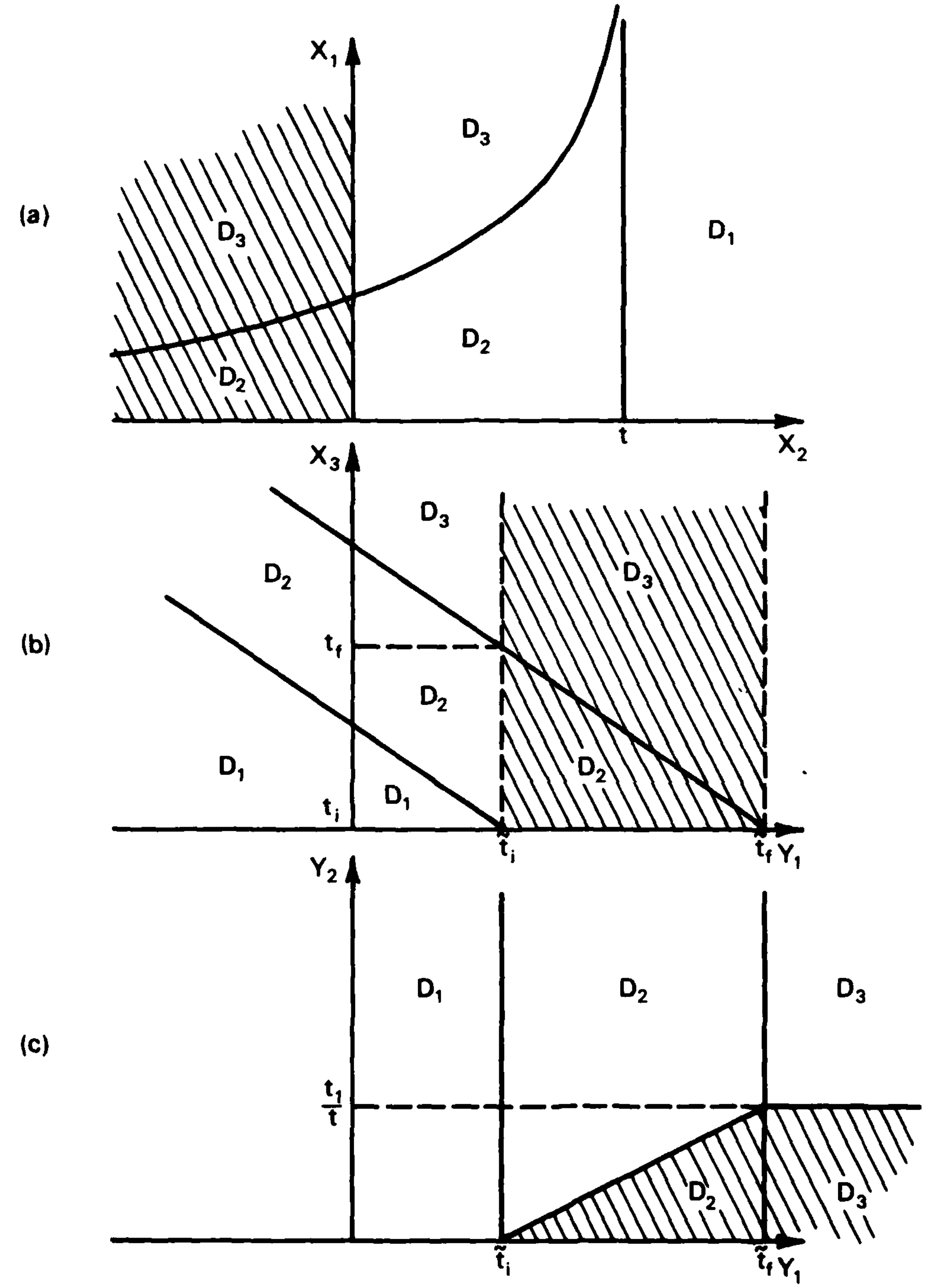

FIG. 6.2. POSITIONS OF SUBSPACE BOUNDARIES FOR THE EXTENDED FORD MODEL. SHADED AREAS DENOTE INITIALLY CRACKED STRUCTURES

(a) (X_2, X_1) space.

(b) (Y_1, X_3) space, $(X_3$ = initial age)

(c) (Y_1, Y_2) space.

where,

$$\bar{t}_{f,y_2} = \min\{\bar{t}_f, \bar{t}_i + y_2 t\}. \tag{6.30}$$

The density function for the time to failure is given by,

$$\begin{aligned}
p_F(t) &= r_1 \exp\{-r_1 t\} P_{Y_1}(\bar{t}_i) + \\
&+ \int_0^{\infty} p_{Y_2}(y_2) \int_{\bar{t}_i}^{\bar{t}_{f,y_2}} r_2(y_1) p_{Y_1}(y_1|y_2) \exp\left\{-r_1 \,.\, (t-(y_1-\bar{t}_i)/y_2) - \right. \\
&\left. - \frac{1}{y_2} \int_{\bar{t}_i}^{y_1} r_2(y_1') dy_1' \right\} dy_1 dy_2 + \\
&+ \int_{\bar{t}_i/t}^{\infty} y_2 p_{Y_2}(y_2) p_{Y_1}(\bar{t}_f|y_2) \exp\left\{-(t-\bar{t}_i/y_2) r_1 - \frac{1}{y_2} \int_{\bar{t}_i}^{\bar{t}_f} r_2(y_1') dy_1' \right\} dy_2
\end{aligned} \tag{6.31}$$

which can be readily verified to be equivalent (but based on a reversed order of integration), to the expression given in equation (6.13).

6.5 Ford's One-Crack Model

Ford[9] initially derived a single parameter model using initiation time as the random variable. This model, referred to here as Ford's "one-crack" model (although strictly *all* models described herein are for a single crack) can be deduced from the equations obtained in Section 6.4 by setting $Y_2 = 1$, i.e.,

$$\begin{aligned}
p_{Y_1}(y_1) &\equiv p_{Y_1}(y_1|Y_2 = 1) \\
&= p_{X_2}(t+\bar{t}_i-y_1),
\end{aligned} \tag{6.32}$$

$$\begin{aligned}
P_S(t) &= \exp\{-r_1 t\} P_{Y_1}(\bar{t}_i) + \\
&+ \int_{\bar{t}_i}^{\min\{\bar{t}_f, \bar{t}_i + t\}} p_{Y_1}(y_1) \exp\left\{-(t+\bar{t}_i-y_1) r_1 - \int_{\bar{t}_i}^{y_1} r_2(y_1') dy_1' \right\} dy_1
\end{aligned} \tag{6.33}$$

and

$$\begin{aligned}
p_F(t) &= r_1 \exp\{-r_1 t\} P_{Y_1}(\bar{t}_i) + \\
&+ \int_{\bar{t}_i}^{\min\{\bar{t}_f, \bar{t}_i + t\}} r_2(y_1) p_{Y_1}(y_1) \exp\left\{-(t+\bar{t}_i-y_1) r_1 - \int_{\bar{t}_i}^{y_1} r_2(y_1') dy_1' \right\} dy_1 + \\
&+ \delta(\bar{t}_f, t+t_i) p_{Y_1}(\bar{t}_f) \exp\left\{-(t-\bar{t}_i) r_1 - \int_{\bar{t}_i}^{\bar{t}_f} r_2(y_1') dy_1' \right\}.
\end{aligned} \tag{6.34}$$

Equation (6.34) is equivalent to equation (4.7) of Ford.

6.6 Inspections

Ford[10] presents a general analysis for the effects of arbitrary inspections on his one-crack model. Here, the effects of perfect inspections, analogous to those modelled in Section 4.4, are included in the two-parameter model.

The effect of an inspection, at ti_j say, is the removal of all structures with,

$$x_1(ti_j - x_2) \geqslant \bar{t}_d - \bar{t}_i \tag{6.35}$$

where $\bar{t}_d$ is the life corresponding to the inspection criterion (equation (4.33)). At time $t > ti_j$, the effect of the inspection can be represented by the removal function,

$$S(\mathbf{x}, ti_j) = H_s(\bar{t}_d - \bar{t}_i + x_1(x_2 - ti_j)) \tag{6.36}$$

or

$$S(\mathbf{y},ti_j) = H_s(\bar{t}_d - y_1 + y_2(t - ti_j)). \tag{6.37}$$

These removal functions can be either inserted in the integrands of the reliability functions or represented by modified integration limits. In terms of the transformed random variables, the effect of the inspection at ti_j can be included by replacing $\bar{t}_f$ in (6.30) by $\bar{t}_f^*$ where,

$$\bar{t}_f^* = \min\{\bar{t}_f, \bar{t}_d + y_2(t - ti_j)\}. \tag{6.38}$$

The third term in (6.31) will exist only when $\bar{t}_f^* = \bar{t}_f$.

6.7 Failure Density for Crack Length

Ford[9] *obtains the failure density* for crack length for the one-crack model. Using the analysis of Section 2.6, this density can be obtained by transforming the integral term in (6.34),

$$\begin{aligned} p_{\mathrm{F}}(t) = {} & r_1(\exp\{-r_1 t\} P_{\mathbf{Y}_1}(a^{-1}(a_i)) + \\ & + \int_{a_i}^{a_f} r_2(a^{-1}(a)) p_{\mathbf{Y}_1} \frac{(a^{-1}(a))}{a'} \exp\left\{ -(t + \bar{t}_i - a^{-1}(a)) r_1 - \int_{a_i}^{a} \frac{r_2(a^{-1}(a^*)) da^*}{a'} \right\} da + \\ & + \delta(\bar{t}_f, t + \bar{t}_i) p_{\mathbf{Y}_1}(a^{-1}(a_f)) \exp\left\{ -(t - \bar{t}_i) r_1 - \int_{a_i}^{a_f} \frac{r_2(a^{-1}(a^*))}{a'} da^* \right\} \end{aligned} \tag{6.39}$$

where $a = a(t)$, $a_i = \mathrm{a}(\bar{t}_i)$, $a_f = a(\bar{t}_f)$ and $a' = da/dt$.

For cracked structures, with $a_i < a < a_f$,

$$p_{\mathrm{A}}(a|t) = \frac{r_2(a^{-1}(a))}{a'} p_{\mathbf{Y}_1}(a^{-1}(a)) \exp\left\{ -(t + \bar{t}_i - a^{-1}(a)) - \int_{a_i}^{a} \frac{r_2(a^{-1}(a^*))}{a'} da^* \right\} \tag{6.40}$$

which is equivalent to equation (4.12) presented by Ford.[9]

6.8 Model Extensions to Include Initial Cracking

As postulated, Ford's assumptions do not allow for the possibility that structures may be cracked at initial time ($t = 0$). Diamond and Payne[3] presented analysis for a population in which *all* structures are initially cracked. In this section, Ford's model is extended to account for such structures which may form only part of the total number of structures. The Diamond and Payne model emerges as a limiting case of this extended model.

A structure that is cracked when $t = 0$, can be considered as having a negative crack initiation time and commences life at $t = 0$ with an "initial age" given by,

$$y_1(0) = \bar{t}_i - x_1 x_2. \tag{6.41}$$

Such structures can be included in the Ford model by extending the range of the random variable $\mathbf{X}_2$ to $(-\infty, \infty)$ so that (6.26) is replaced by,

$$0 < y_2 < \infty, \qquad \bar{t}_i \leqslant y_1 \leqslant \bar{t}_f, \tag{6.42}$$

(6.28) by,

$$P_S^2(\mathbf{y},t) = \exp\left\{ -\max\{0, t - (y_1 - \bar{t}_i)/y_2\} r_1 - \frac{1}{y_2} \int_{\max\{\bar{t}_i, y_1 - y_2 t\}}^{y_1} r_2(y_1') dy_1' \right\} \tag{6.43}$$

and (6.29) by

$$P_S(t) = \exp\{-r_1 t\}P_{Y_1}(\bar{t}_i)+$$
$$+\int_0^{\infty} p_{Y_2}(y_2)\int_{\bar{t}_i}^{\bar{t}_f} p_{Y_1}(y_2|y_1)\exp\left\{-\max\{0,t-(y_1-\bar{t}_i)/y_2\}\}r_1 - \right.$$
$$\left. -\frac{1}{y_2}\int_{\max\{\bar{t}_i,y_1-y_2 t\}}^{y_1} r_2(y_1')dy_1'\right\}dy_1 dy_2. \tag{6.44}$$

Corresponding equations for the derived reliability functions follow trivially from equation (6.44).

6.8.1 Initial Age as a Random Variable

The model based on equation (6.44) requires that initially-cracked structures are represented by deduced negative initiation times. A more convenient random variable for such structures is the initial age defined by equation (6.41). Using $\mathbf{X}_3$ to denote the age of a structure at $t = 0$, a structure which is cracked at $t = 0$, has age $\mathbf{Y}_1$ at time t given by,

$$\mathbf{Y}_1 = \mathbf{X}_3+\mathbf{X}_1 t \tag{6.45}$$

where

$$\mathbf{X}_3 = \bar{t}_i-\mathbf{X}_1\mathbf{X}_2. \tag{6.46}$$

The definition of initial age for an uncracked structure is somewhat arbitrary. One definition which allows a one-to-one mapping between initial age and crack initiation time is,

$$\mathbf{Y}_1 = \mathbf{X}_3+t \tag{6.47}$$

where

$$\mathbf{X}_3 = \bar{t}_i-\mathbf{X}_2. \tag{6.48}$$

This definition amounts to the assumption that the variation in crack initiation time arises from variations in pre-aging of the structure rather than variations in the damage rate (as implied by equation (6.20), which assumes all uncracked structures have zero age at $t = 0$).

The reliability model can be expressed in terms of the random variables, age ($\mathbf{Y}_1$) and comparative crack growth rate ($\mathbf{Y}_2$),

$$\mathbf{Y}_2 = \mathbf{X}_1, \tag{6.49}$$

$$\mathbf{Y}_1 = \begin{cases} \mathbf{X}_3+t, & \mathbf{Y}_1 < \bar{t}_i \\ \mathbf{X}_3+\mathbf{Y}_2 t, & \mathbf{Y}_1 \geqslant \bar{t}_i \end{cases} \tag{6.50}$$

so that for structures in D_1,

$$p_{Y_1}(y_1) = p_{X_3}(y_1-t) \tag{6.51}$$

and for those in D_2

$$p_{Y_1}(y_1|y_2) = p_{X_3}(y_1-y_2 t). \tag{6.52}$$

Equation (6.44) can then be used to determine the probability of survival.

6.8.2 The Initial Crack Model of Diamond and Payne

The initial crack model developed by Diamond and Payne can be deduced from the equations developed in the previous section. Assuming that *all* structures are initially cracked, it follows that D_1 is empty and that D_2 is defined by,

$$0 < y_2 < \bar{t}_i/t, \qquad \bar{t}_i+y_2 t < y_1 < \bar{t}_f \tag{6.53}$$

(see Fig. 6.2).

The equation for the probability of survival, (6.44), becomes, using (6.52),

$$P_s(t) = \int_0^{\bar{t}_i/t} p_{X_1}(y_2) \int_{\bar{t}_i+y_2t}^{\bar{t}_f} p_{X_3}(y_1-y_2t)\exp\left\{-\frac{1}{y_2}\int_{y_1-y_2t}^{y_1} r_2(y_1')dy_1'\right\}dy_1dy_2. \quad (6.54)$$

Transforming back to the original random variables X_1 and X_3 and reversing the order of integration,

$$P_s(t) = \int_{\bar{t}_i}^{\bar{t}_f} p_{X_3}(x_3) \int_0^{(\bar{t}_f-x_3)/t} p_{X_1}(x_1)\exp\left\{-\frac{1}{x_1}\int_{x_3}^{x_3+x_1t} r_2(y_1')dy_1'\right\}dx_1dx_3 \quad (6.55)$$

or if comparative fatigue life (denoted in this section by X_1^*) is used instead of comparative crack growth rate,

$$P_s(t) = \int_{\bar{t}_i}^{\bar{t}_f} p_{X_3}(x_3) \int_{t/(\bar{t}_f-x_3)}^{\infty} p_{X_1^*}(x_1^*)\exp\left\{-x_1^*\int_{x_3}^{x_3+t/x_1^*} r_2(y_1')dy_1'\right\}dx_1^*dx_3. \quad (6.56)$$

The corresponding expression for the risk of static failure by fatigue is,

$$P_s(t)r_s(t) = \int_{\bar{t}_i}^{\bar{t}_f} p_{X_3}(x_3) \int_{t/(\bar{t}_f-x_3)}^{\infty} r_2(x_3+t/x_1^*)p_{X_1^*}(x_1^*)\exp\left\{-x_1^*\int_{x_3}^{x_3+t/x_1^*} r_2(y_1')dy_1'\right\}dx_1^*dx_3 \quad (6.57)$$

which if the exponential term is ignored is the same as the approximate expression developed by Diamond and Payne.

Further details of the initial crack model are discussed, as part of the documentation for the NERF computer program, by Mallinson and Graham.[13]

7. A THREE-PARAMETER MODEL

A natural extension to the model described in Section 6.8 is to include a random variable representing strength variation. The resulting three-parameter model has sufficient generality to encompass, as special cases, all the models discussed in this report and is presented here using notation which permits a simple tabulation of the special cases. All these models can be deduced from this tabulation leaving the procedures outlined in Section 3.6 for the development of new models as the need arises.

7.1 Assumptions and Model Equations

7.1.1 Basic Random Variables

The model is assumed to depend on three random variables two of which (X_1, X_2) are parameters in the equation for strength decay as a function of time (e.g. equation (4.9)) for cracked structures. The third random variable, X_3, is a parameter affecting the relationship between the strength of a given structure and some known median strength for the population (e.g. x_2 in (4.10)).

The basic random variables will be specified with no further precision. This allows the three-parameter model to be defined in terms of its transformed random variables while leaving the transformations unspecified, facilitating reduction to a wider class of less general models.

7.1.2 Transformed Random Variables

The transformed random variables are: age (Y_1), comparative crack growth rate (Y_2) and virgin strength (Y_3). They are assumed to bear the following functional relationships with the basic random variables.

$$\mathbf{Y}_1 = \mathbf{Y}_1(\mathbf{X}_1,\mathbf{X}_2,t), \tag{7.1}$$

$$\mathbf{Y}_2 = \mathbf{Y}_2(\mathbf{X}_1,\mathbf{X}_2), \tag{7.2}$$

$$\mathbf{Y}_3 = \mathbf{Y}_3(\mathbf{X}_3). \tag{7.3}$$

For, given time, the joint density function for $\mathbf{Y}_1$ and $\mathbf{Y}_2$ can be written in the form,

$$p_{\mathbf{Y}_1,\mathbf{Y}_2}(y_1,y_2) = p_{\mathbf{Y}_2}(y_2)p_{\mathbf{Y}_1}(y_1|y_2); \tag{7.4}$$

for uncracked structures $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are independent. The density functions $p_{\mathbf{Y}_1}(y_1|y_2)$, $p_{\mathbf{Y}_2}(y_2)$ and $p_{\mathbf{Y}_3}(y_3)$ can be derived from the density functions for $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ using equation (3.62).

7.1.3 Initial Age

For any structure, the age when $t = 0$ is called the initial age (y_0) where,

$$y_0 = y_1(x_1,x_2,0). \tag{7.5}$$

It is assumed that the time history equations are defined so that for given time y_0 can be alternatively expressed in terms of y_1 and y_2, viz.,

$$y_0 = y_0(y_1,y_2,t) \tag{7.6}$$

and y_1 in terms of y_0 and y_2,

$$y_1 = y_1(y_0,y_2,t). \tag{7.7}$$

For a given population, initial age may be limited so that,

$$y_{0,1} \leqslant y_0 \leqslant y_{0,2}. \tag{7.8}$$

7.1.4 Model Equations and Subspace Boundaries

For structures in D_1,

$$r_1 = r_1(y_3) \tag{7.9}$$

and

$$R_{\min} < y_3; \qquad y_1 < \bar{t}_i. \tag{7.10}$$

For cracked structures in D_2,

$$r_2 = r_2(y_3\psi(y_1)) \tag{7.11}$$

and

$$R_{\min} < y_3; \qquad 0 \leqslant y_2 \leqslant y_{2,2}; \qquad y_{1,1} \leqslant y_1 \leqslant y_{1,2} \tag{7.12}$$

where

$$y_{1,1} = \max\{\bar{t}_i, y_1(y_{0,1},y_2)\} \tag{7.13}$$

$$y_{1,2} = \min\{\bar{t}_{f,y_3}, y_1(y_{0,2},y_2)\} \tag{7.14}$$

and $y_{2,2}$ is defined by,

$$y_1(y_{0,1},y_{2,2},t) = \bar{t}_{f,y_3}. \tag{7.15}$$

Fatigue life limiting, or the effects of perfect inspections are included in the parameter $\bar{t}_{f,y_3}$ where,

$$\bar{t}_{f,y_3} = \min\{\bar{t}_1^{*}, \psi^{-1}(R_{\min}/y_3)\} \tag{7.16}$$

and

$$\bar{t}_1^{*} = \min\{\bar{t}_f, \bar{t}_{f}(y_2,t,ti_j)\}. \tag{7.17}$$

The function $\bar{t}_f(y_2,t,ti_j)$ represents the effect of an inspection at $t = ti_j$.

7.2 An Integral Expression for $P_S(t)$

Using the model equations, the following equations can be derived,

$$p_S^1(\mathbf{y},t) = \exp\{-r_1(y_3)t\} \tag{7.18}$$

and

$$\begin{aligned} P_S^2(\mathbf{y},t) &= \exp\left\{-t_1 r_1(y_3) - \int_{t_1}^{t} r_2(y_3\psi(x_1,x_2,t')dt'\right\} \\ &= \exp\left\{-t_1 r_1(y_3) - \int_{y_1(t_1)}^{y_1} \frac{r_2(y_3\psi(y_1'))}{dy_1'/dt} dy_1'\right\} \\ &= H_y(y_1,y_2,y_3,t), \end{aligned} \tag{7.19}$$

where

$$t_1 = \max\{0,t{:}y_1(x_1,x_2,t) = \bar{l}_i\}. \tag{7.20}$$

The expression for the probability of survival is,

$$\begin{aligned} P_S(t) = &\int_{R_{min}}^{\infty} \exp\{-r_1(y_3)t\} p_{Y_3}(y_3) dy_3 P_{Y_1}(\bar{l}_i) + \\ &+ \int_{R_{min}}^{\infty} p_{Y_3}(y_3) \int_0^{y_{2,2}} p_{Y_2}(y_2) \int_{y_{1,1}}^{y_{1,2}} p_{Y_1}(y_1|y_2) H_y(\mathbf{y},t) dy_1 dy_2 dy_3. \end{aligned} \tag{7.21}$$

7.3 Risk Rate Equations

The expressions for virgin risk, the risk of static fracture by fatigue and the risk of fatigue life exhaustion, obtained by differentiating (7.21) with respect to time are given by

$$P_S(t) r_v(t) = \int_{R_{min}}^{\infty} r_1(y_3) \exp\{-r_1(y_3)t\} p_{Y_3}(y_3) dy_3 P_{Y_1}(\bar{l}_i), \tag{7.22}$$

$$P_S(t) r_s(t) = \int_{R_{min}}^{\infty} p_{Y_3}(y_3) \int_0^{y_{2,2}} p_{Y_2}(y_2) \int_{y_{1,1}}^{y_{1,2}} r_2(y_1) p_{Y_1}(y_1|y_2) H_y(\mathbf{y},t) dy_1 dy_2 dy_3 \tag{7.23}$$

and

$$\begin{aligned} P_S(t) r_f(t) = &\int_{R_{min}}^{R_{min}/\psi(\bar{l}_f^*)} p_{Y_3}(y_3) \int_{y_{2,t}}^{y_{2,2}} p_{Y_2}(y_2) \left.\frac{\partial y_1}{\partial t}\right|_{\bar{l}_R} p_{Y_1}(\bar{l}_R|y_2) H_y(\bar{l}_R,y_2,y_3,t) dy_2 dy_3 + \\ &+ \delta(\bar{l}_f,\bar{l}_f^*) \int_{R_{min}/\psi(\bar{l}_f)}^{\infty} p_{Y_3}(y_3) \int_{y_{2,t}}^{y_{2,2}} p_{Y_2}(y_2) \left.\frac{\partial y_1}{\partial t}\right|_{\bar{l}_f} p_{Y_1}(\bar{l}_f|y_2) H_y(\bar{l}_f,y_2,y_3,t) dy_2 dy_3 \end{aligned} \tag{7.24}$$

where

$$\bar{l}_R = \psi^{-1}(R_{min}/y_2) \tag{7.25}$$

and $y_{2,t}$ is given by,

$$y_1(y_{0,2},y_{2,t},t) = \bar{l}_{f,y_3}. \tag{7.26}$$

7.4 Reduction to Simpler Models

Equations (7.21) to (7.25) define the probability of survival and risk rates for any three-parameter model satisfying the assumptions in Section 7.1. In particular, all the models described in this report can be considered as simplifications of this three-parameter model.

Table 7.1 details the functional forms required to effect the simplifications for models grouped in the following manner. The column headed "Ford (including extension)" incorporates all the models described in Chapter 6 that use crack growth rate and initiation time as the basic random variables associated with age. The term "initial crack" identifies models that follow Diamond and Payne[3] and use initial age and comparative fatigue life as the age random variables. The third column in the table details the two-parameter model described in Chapter 4. Hooke's model (Chapter 5), is also covered by the entries in this column if $\bar{R}_0$ is set equal to 1 and X_3 interpreted as virgin strength.

Note that the initial crack model includes the strength random variable whereas the extended Ford model does not. This follows the model definitions of the originating authors and this random variable can be included or neglected by using the relevant table entries from either model. In this sense, the first two columns describe the same model, the essential difference between the table entries being the definition of the basic random variables.

TABLE 7.1

Functional Forms Required to Reduce the Three-parameter Model to the Extended Ford, Initial Crack (after Diamond and Payne) and Payne *et al.* Two-parameter Models

Model	Ford (including extension)	Initial crack	Payne *et al.* (two-parameter)
X_1	Crack growth rate	Comparative fatigue life	Comparative fatigue life
X_2	Initiation time	Initial age	n.a.
X_3	n.a.	Relative residual strength	Relative residual strength
Y_1 (age)	$\bar{t}_i t/X_2;\ Y_1 < \bar{t}_i$ $\bar{t}_i + Y_2(t - X_2);$ $Y_1 \geqslant \bar{t}_i$	$X_2 + t;\ Y_1 < \bar{t}_i$ $X_2 + Y_2 t;$ $Y_1 \geqslant \bar{t}_i$	t/X_1
Y_2 (crack growth rate)	X_1	$1/X_1$	n.a.
Y_3 (virgin strength)	n.a.	$\bar{R}_0 X_3$	$\bar{R}_0 X_3$
$p_{Y_1}(y_1 \mid y_2)$	$p_{X_2}\left(\dfrac{t-(y_1-\bar{t}_i)}{y_2}\right)\Big/ y_2$	$p_{X_2}(y_1 - y_2 t)$	$p_{X_1}(t/y_1)t/y_1^2$
$p_{Y_2}(y_2)$	$p_{X_1}(y_2)$	$p_{X_1}(1/y_2)/y_2^2$	n.a.
$p_{Y_3}(y_3)$	n.a.	$p_{X_3}(y_3/\bar{R}_0)/\bar{R}_0$	$p_{X_3}(y_3/\bar{R}_0)/\bar{R}_0$
$P_{Y_1}(\bar{t}_i)$	$\bar{P}_{X_2}(t)$	0	$P_{X_1}(t/\bar{t}_i)$
$y_0(y_1, y_2, t)$	$y_1 - y_2 t$	$y_1 - y_2 t$	0
$y_1(y_0, y_2, t)$	$y_0 + y_2 t$	$y_0 + y_2 t$	n.a.
$\bar{t}_{ti}$	$\bar{t}_d + y_2(t - ti_j)$	$\bar{t}_d + y_2(t - ti_j)$	$\bar{t}_d t/ti_j$
t_1	$t - (y_1 - \bar{t}_i)/y_2$	$t - (y_1 - \bar{t}_i)/y_2$	$t\bar{t}_i/y_1$

Table 7.2 details the integration limits appearing in equations (7.21) to (7.25). The extended Ford model has been subdivided into a model with no initial cracks (the original Ford[11] model), and one allowing for some of the population of structures to be initially cracked. The initial crack model assumes *all* structures are initially cracked. Bearing in mind the comments in the penultimate paragraph, the first three columns of Table 7.2 can be used with either of the first two columns in Table 7.1 to generate the expressions relevant to the desired degree of initial cracking. Note that the effect of the strength random variable is detailed in column 3 for the initial crack model only.

All the entries in Table 7.2 are relevant for fatigue life limiting which can be removed by setting $\bar{t}_f = \infty$, and/or $R_{min} = 0$ appropriately.

TABLE 7.2

Integration Limits Required to Reduce the Three-parameter Model to the Extended Ford, Initial Crack and Payne *et al.* Two-parameter Models

Model	Ford (including extension)		Initial crack	Payne *et al.*
	No initial cracks	Some initial cracks		
$y_{0,1}$	0	0	$\bar{t}_i$	0
$y_{0,2}$	$\bar{t}_i$	$\bar{t}_f$	$\bar{t}_f$	0
$y_{1,1}$	$\bar{t}_i$	$\bar{t}_i$	$\max\{\bar{t}_i, \bar{t}_i + y_2 t\}$	$\bar{t}_i$
$y_{1,2}$	$\min\{\bar{t}_f, \bar{t}_i + y_2 t\}$	$\bar{t}_f$	$\bar{t}_{f,y_3}$	$\bar{t}_{f,y_3}$
$y_{2,2}$			$(\bar{t}_{f,y_3} - \bar{t}_i)/t$	n.a.
y_f	$\bar{t}_i/t$	0	0	n.a.

8. CONCLUSIONS

The method developed in Chapter 3 for the generation of reliability models has been demonstrated, by example, to be successful in producing correct forms of the reliability functions relevant to models used previously at ARL. In particular, the systematic application of a unified method has led to an appreciation of differences between the models and in some cases, correction of previously published expressions.

The two-parameter model developed in Chapter 4 using the assumptions of the Payne *et al.*[1-4] model was shown to yield expressions presented by Payne and Graham[4] for the risks of static failure by fatigue and fatigue fracture. The latter risk was shown to have two contributions; that arising from an imposed limit on fatigue life and a contribution arising from a prescribed minimum strength value. The model developed in Chapter 4 also accounts for failures of uncracked structures whose strengths have not been reduced below their virgin values. An expression for the resulting virgin risk was obtained and the expressions for the other reliability functions account for this effect. This treatment of virgin risk mirrors that of Yang and Trapp.[14]

Expressions for the probability density for strength and the failure density of strength were obtained and the latter was shown to reduce to an approximate expression derived by Diamond and Payne.[3]

The two-parameter model described by Hooke, although using a slightly different definition for the basic random variables, was shown to be basically the same as the Payne *et al.* model with the exception of the assumptions regarding fatigue life limiting. Although Hooke's equations for the probability of survival and the risk of static failure by fatigue do not account for any

imposed limit on fatigue life, Hooke[8] derives an expression for the risk of fatigue life exhaustion which accounts only for fatigue life limiting. (No minimum value of strength is prescribed.) In this sense, Hooke's equations are inconsistent.

An examination of Hooke's proposal[8] to isolate the failures of structures that are weakened by fatigue cracking and fail by a load less than their virgin strengths concluded that the approach based on taking the algebraic difference between two probabilities of survival, one with fatigue cracking and one without, overestimated such failures. Correct isolation can be obtained by defining the basic risk rates used in the model appropriately and generating the corresponding reliability functions using the techniques described in Chapter 3.

Using the basic random variables defined by Ford,[9,11] the equivalence between the method described herein and his technique based on solving a partial differential continuity equation for the probability density for crack length was established. Transformation of the random variables led to forms of the reliability functions which could be readily compared with the Payne *et al.* functions. By introducing initial age as a random variable, Ford's model was extended to account for structures that are initially cracked. This extended model encompasses the initial crack model suggested by Diamond and Payne[3] for the analysis of the reliability of a population of structures all of which commence life cracked.

The three-parameter model presented in Chapter 7 encompasses all the models described in this report; a tabulation of the simplifications relevant to each model was presented. For these models, this tabulation fulfills one of the objectives of the investigation by providing a basis for the complete specification for the numerical evaluation of the reliability functions. For other models, or for variations (such as different inspection procedures) of the tabulated models, the method described in Chapter 3 provides the fundamental means for fulfilling this objective.

NOTATION

a or $a(t)$	Crack length, function of time
a'	da/dt
a_i	Crack length immediately after initiation
a_f	Crack length corresponding to imposed fatigue life limit
A	Random variable representing crack length
c_i	Scaling constant for the ith random variable—used in transformations
D_k	kth subspace of sample space for the random variables describing the variation in the reliability model. Each subspace represents a distinct time zone in the fatigue process. For the models described herein, three subspaces are used as defined below.
D_1	Subspace containing uncracked structures
D_2	Subspace containing cracked structures
D_3	Subspace containing structures which have passed a failure criterion
$f(a)$	Crack growth rate function, equation (6.1)
$dF_k(\mathbf{x},t)$	$P_{\mathbf{S}}^k(\mathbf{x},t)p_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$, see equation (3.3)
$f(z)$	Function of z used in the derivation of the failure density of a structural characteristic
$F(x_1,y_1,t)$	Homogeneous function defining the transformation from x_1 to y_1, see equation (3.71)
$g(z)$	Function of z used in the derivation of the density function for a structural characteristic
$H(\mathbf{x},t)$	(or $H(x_1, x_2, \ldots, x_M, t)$). Loss factor for **x**, see equation (3.19)
$H_s(t)$	Heaviside step function
$H_y(\mathbf{y},t)$	Loss factor for **y**
i or i (subscript)	ith random variable or initiation, depending on context
$\mathrm{I}_1(x_2,t)$	Inner integration over x_1, see equation (3.31)
k	Subscript or superscript, identifies subspace
K	Number of subspaces in model
l_f	Average frequency of load application
L	Random variable representing applied load
L	Sample value for **L**
M	Number of random variables in model
n (superscript)	Denotes a definition relevant to a finite population of structures
$P(A)$	Probability of A
$P_{\mathbf{A}}(a)$	Probability that $\mathbf{A} \leq a$

$\bar{P}_{\mathbf{A}}(a)$	Probability that $\mathbf{A} > a$, $(=1-P_{\mathbf{A}}(a))$
$p_{\mathbf{A}}(a)$	Density function for crack length
$P_{det}(ti_j)$	Probability of detection at the inspection at $t = ti_j$
$P_{\mathbf{F}}(t)$	Probability of failure, $(=P(\mathbf{F} \leqslant t))$
$p_{\mathbf{F}}(t)$	Density function for the time to failure
$p_{\mathbf{F},\mathbf{Z}}(t,z)$	Joint density function for time of failure (F) and structural characteristic (Z)
$\bar{P}_{\mathbf{L}}(R)$	Probability that an applied load exceeds strength R, $(=P(\mathbf{L} > R))$
$p_{\mathbf{R}}(\mathbf{R}\|t)$	Failure density for strength
$P_{\mathbf{S}}(t)$	Probability of survival, $(=P(\mathbf{F} > t))$
$P^-_{\mathbf{S}}(ti_j)$	Probability of survival immediately before an inspection
$P^+_{\mathbf{S}}(ti_j)$	Probability of survival immediately after an inspection
$P^k_{\mathbf{S}}(\mathbf{x},t)$	(or $P^k_{\mathbf{S}}(\mathbf{y},t)$) Probability of survival for structures with $\mathbf{X} = \mathbf{x}$ (or $\mathbf{Y} = \mathbf{y}$), in the subspace D_k
$P_{\mathbf{X}}(\mathbf{x})$	Joint density function for the random variables, $\mathbf{X}$
$p_{\mathbf{Y}_1}(y_1\|y_2)$	Conditional density for $\mathbf{Y}_1$ given $\mathbf{Y}_2 = y_2$
$p_{\mathbf{Y}_1,\mathbf{Y}_2}(y_1,y_2)$	Joint density for $\mathbf{Y}_1$ and $\mathbf{Y}_2$
$p_{\mathbf{Z}}(z\|t)$	Failure density for the structural characteristic, $\mathbf{Z}$
$p_{\mathbf{Z}}(z\|\mathbf{F} > t)$	Conditional density for $\mathbf{Z}$ given survival to t. Loosely referred to as the density for $\mathbf{Z}$.
$r(t)$	Risk rate at time t
$r_a(t)$	Additional risk
r_c	Constant risk (additional)
$r_f(t)$	Risk of fatigue life exhaustion at time t
$r_k(\mathbf{x},t)$	Risk rate in the kth subspace, D_k
$r_s(t)$	Risk of static failure by fatigue at time t
$r_u(t)$	Risk of ultimate failure, see equation (5.23)
$r_v(t)$	Virgin risk at time t
$r_1(\mathbf{x})$	Risk rate for uncracked structures with $\mathbf{X} = \mathbf{x}$
$r_1(\mathbf{y})$	Risk rate for uncracked structures with $\mathbf{Y} = \mathbf{y}$
$r_2(\mathbf{x},t)$	Risk rate for cracked structures with $\mathbf{X} = \mathbf{x}$
$r_2(\mathbf{y},t)$	Risk rate for cracked structures with $\mathbf{Y} = \mathbf{y}$
$r''(\mathbf{x},t)$	Risk rate for cracked structures as a result of loads less than virgin strength
$\mathbf{R}$	Random variable representing strength
R	Strength
$\tilde{R}$	Median strength
R_0	Virgin strength
$\tilde{R}_0$	Median virgin strength
R_{min}	Minimum value of strength
$\mathbf{S}$	Random variable representing survival

$S(\mathbf{x},ti_j)$	Removal function representing the effect of an inspection at time ti_j
t	Time
t_0	Initial time
t'	Dummy integration variable for time
t_k	Time of transition from subspace D_k to D_{k+1}
$\tilde{t}_d$	Median time corresponding to inspection criterion
$\tilde{t}_i$	Median initiation time
t_f	Fatigue life limit ($=t_2$ below)
$\tilde{t}_f$	Median fatigue life limit
$\tilde{t}_{f,x_2}$	Age limit dependent on parameter x_2. Similarly $\tilde{t}_{f,y_2}$, $\tilde{t}_{f,y_3}$.
$\tilde{t}_f^*$	Time limit representing the effect of a perfect inspection
$\tilde{t}_1$	$\tilde{t}_f - \tilde{t}_i$
ti_j	Time of jth inspection
u (superscript)	Reliability function with virgin strength preserved
w (superscript)	Reliability function for weakened structures failing by loads less than the virgin strengths
$\mathbf{X}_i$	ith random variable in model. Meanings of the particular variables change between chapters, or models, but are consistent within a chapter
x_i	Sample value of ith random variable
$x_{i,1}$	Value of x_i at boundary between D_1 and D_2
$x_{i,2}$	Value of x_i at boundary between D_2 and D_3
$\mathbf{X}$	Vector of random variables
$\mathbf{x}$	Vector of sample values of x_i
$\mathbf{Y}_i$	ith transformed random variable
$\mathbf{Y}_1$	Random variable representing age
y_0	Initial age
$y_{i,k}$	Value of y_i at boundary between D_k and D_{k+1}
$\psi(y_1)$	Median relative strength decay as a function of age
′ (prime)	(1) Used to denote new or different form of existing function (2) Denotes dummy integration variable (3) Denotes differentiation with respect to time

$S(\mathbf{x},ti_j)$	Removal function representing the effect of an inspection at time ti_j
t	Time
t_0	Initial time
t'	Dummy integration variable for time
t_k	Time of transition from subspace D_k to D_{k+1}
$\bar{t}_d$	Median time corresponding to inspection criterion
$\bar{t}_i$	Median initiation time
t_f	Fatigue life limit ($=t_2$ below)
$\bar{t}_f$	Median fatigue life limit
$\bar{t}_{f,x_2}$	Age limit dependent on parameter x_2. Similarly $\bar{t}_{f,y_2}$, $\bar{t}_{f,y_3}$.
$\bar{t}_f^{\dagger}$	Time limit representing the effect of a perfect inspection
$\bar{t}_1$	$\bar{t}_f - \bar{t}_i$
ti_j	Time of jth inspection
u (superscript)	Reliability function with virgin strength preserved
w (superscript)	Reliability function for weakened structures failing by loads less than the virgin strengths
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$\mathbf{X}$	Vector of random variables
$\mathbf{x}$	Vector of sample values of x_i
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y_0	Initial age
$y_{i,k}$	Value of y_i at boundary between D_k and D_{k+1}
$\psi(y_1)$	Median relative strength decay as a function of age
′ (prime)	(1) Used to denote new or different form of existing function (2) Denotes dummy integration variable (3) Denotes differentiation with respect to time

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